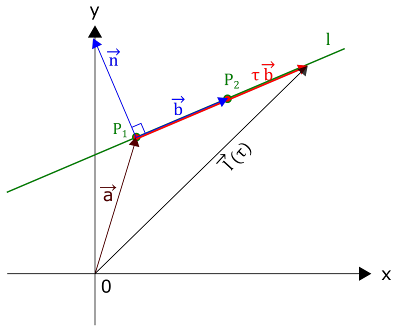
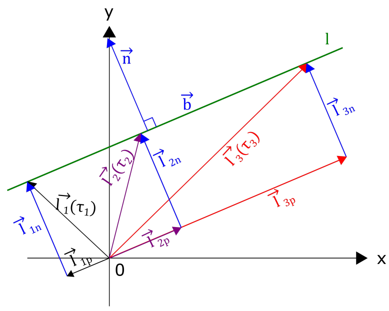
Distance between point and line in a plane



Besides point-direction form: $\vec{l}(\tau) = \vec{a} + \tau\vec{b}$(with a being any point of the line and $\vec{b}$ a vector with the same direction as line) one can also express line in point-normal form - just like needle of a compass not only determines north and south but also east and west. To calculate normal form we apply dot product with normal vector $\vec{n}$ on each side of the point-direction equation:

$$\vec{n} \cdot \vec{l}(\tau) = \vec{n} \cdot \vec{a} + \underbrace{\tau(\vec{n} \cdot \vec{b})}_{=0}$$

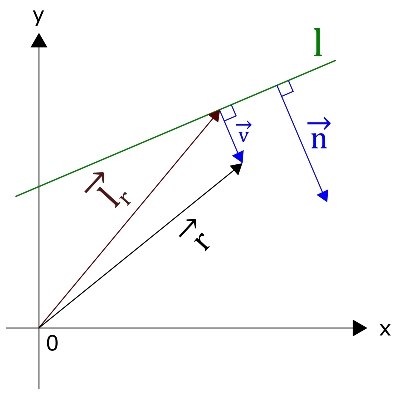


Expression on the left side of previous equation $\vec{n} \cdot \vec{l}(\tau)$ always has the same value for any point of the line and we assign it letter d to express it as a constant value. This might seem strange at first that every vector to any point of the line would have the same dot product result with normal vector but after splitting these vectors in vector components that are parallel and perpendicular to the line one can easily see the reason for it - perpendicular component always stays the same, it is only parallel one that changes its length:

$$\vec{n} \cdot \vec{l}_1(\tau_1) = \vec{n} \cdot (\vec{l}_{1p} + (\vec{l}_{1n}) = \underbrace{\vec{n} \cdot \vec{l}_{1p}}_{=0} + \vec{n} \cdot \vec{l}_{1n} = \vec{n} \cdot \vec{l}_{1n} = d$$

$$\vec{n} \cdot \vec{l}_2(\tau_2) = \vec{n} \cdot (\vec{l}_{2p} + (\vec{l}_{2n}) = \underbrace{\vec{n} \cdot \vec{l}_{2p}}_{=0} + \vec{n} \cdot \vec{l}_{2n} = \vec{n} \cdot \vec{l}_{2n} = d$$

$$\vec{n} \cdot \vec{l}_3(\tau_3) = \vec{n} \cdot (\vec{l}_{3p} + (\vec{l}_{3n}) = \underbrace{\vec{n} \cdot \vec{l}_{3p}}_{=0} + \vec{n} \cdot \vec{l}_{3n} = \vec{n} \cdot \vec{l}_{3n} = d$$



With this we now have a final representation for our point-normal equation of a line:

$$\vec{n} \cdot \vec{a} = d$$

Analogue to standard equation of a plane we can calculate standard equation of a line if we know values of $\vec{n}$ and d - resulting equation is of type $\underbrace{A}_{n_x} x + \underbrace{B}_{n_y} y + \underbrace{C}_{-d} = 0$.

Using these definitions we can now derive a formula for calculating distance between point and a line.
Let $\vec{r}$ be vector to a random point outside of line l. We name $\vec{l}_r$ as vector of the point of the line that is nearest to r. Vector $\vec{v} = \vec{r} - \vec{l}_r$ has length of distance between point r and line l. With normal vector $\vec{n}$ of line l and using $\vec{r} = \vec{l}_r + \vec{v}$ we can apply dot product with $\vec{n}$ on both sides of the equation:

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{l}_r + \vec{n} \cdot \vec{v}$$

$$\vec{n} \cdot \vec{r} = d + |\vec{n}||\vec{v}| \underbrace{\cos\alpha}_{=1}$$

$$|\vec{v}| = \frac{\vec{n} \cdot \vec{r} - d}{|\vec{n}|}$$

Length of $\vec{v}$ is distance between r and l.
How do we find this distance where line is defined through two points and their coordinates: $\vec{p}_1 = [x_1, y_1]^T$ and $\vec{p}_2 = [x_2, y_2]^T$?
Difference between these two vectors is direction vector of the line $\vec{b} = \vec{p}_2 - \vec{p}_1 = [x_2 - x_1, y_2 - y_1]^T$. To construct normal vector from $\vec{b}$ we swap its components and multiply one of them by -1[1]: $\vec{n} = [y_1 - y_2, x_2 - x_1]^T$. To calculate value of d of point-normal form we perform dot product between $\vec{p}_1$ and $\vec{n}$:

$$d = \vec{p}_1 \cdot \vec{n} = \cancel{x_1 y_1} - x_1 y_2 + y_1 x_2 - \cancel{y_1 x_1} = y_1 x_2 - x_1 y_2$$

[1] see solution of question 4 of "Control questions on vector geometry" on page 10 of this document

If we search for distance of point $r = (r_x, r_y)^T$ we can set the values in distance formula:

$$|\vec{v}| = \frac{\vec{n} \cdot \vec{r} - d}{|\vec{n}|} = \frac{\begin{bmatrix} y_1 - y_2 \\ x_2 - x_1 \end{bmatrix} \cdot \begin{bmatrix} r_x \\ r_y \end{bmatrix} - (y_1 x_2 - x_1 y_2)}{\sqrt{(y_1 - y_2)^2 + (x_2 - x_1)^2}}$$

When expressed through abbreviations of standard form of a line we get:

$$|\vec{v}| = \frac{\vec{n} \cdot \vec{r} - d}{|\vec{n}|} = \frac{n_x r_x + n_y r_y - d}{\sqrt{n_x^2 + ny^2}} = \frac{Ar_x + Br_y + C}{\sqrt{A^2 + B^2}}$$