

Practice material: vector math

Using the given pairs of vectors calculate:

- a) their sum $\vec{u} + \vec{v}$ and difference $\vec{u} - \vec{v}$,
- b) length $|\vec{u}|$ and $|\vec{v}|$,
- c) unit vectors \hat{u}_u and \hat{u}_v ,
- d) distance between their end points $|\vec{u} - \vec{v}|$,
- e) dot product $\vec{u} \cdot \vec{v}$,
- f) angle α between \vec{u} and \vec{v} ,
- g) cross-product $\vec{w} = \vec{u} \times \vec{v}$,
- h) angle β between \vec{w} and positive z -axis,
- i) an equation for a line l through the endpoints of both vectors,
- j) point of intersection P_0 of line l through the plane $x - y$,
- k) points P_1 and P_2 on line l , that are in distance of 17 units from endpoint of \vec{u} ,
- l) plane \mathcal{P} with $\vec{u} \in \mathcal{P}$ and $\vec{v} \perp \mathcal{P}$ in standard form through point-normal form,
- m) distance δ_P of \mathcal{P} from the origin,
- n) intersection point P_I of y -axis and \mathcal{P} and
- o) intersection line l_I between plane \mathcal{P} and $x - y$ -plane in form of $y = ax + b$.

- 1) $\vec{u} = [4, 5, 2]^T, \vec{v} = [4, 3, 1]^T$
- 2) $\vec{u} = [4, -4, 2]^T, \vec{v} = [2, 8, 1]^T$
- 3) $\vec{u} = [7, -4, 5]^T, \vec{v} = [7, 3, -8]^T$
- 4) $\vec{u} = [5, 5, 2]^T, \vec{v} = [8, -1, 1]^T$
- 5) $\vec{u} = [-5, 8, 2]^T, \vec{v} = [-9, -4, -7]^T$
- 6) $\vec{u} = [2, -8, -6]^T, \vec{v} = [2, -6, 8]^T$
- 7) $\vec{u} = [3, 5, 7]^T, \vec{v} = [4, -1, -7]^T$
- 8) $\vec{u} = [4, 4, 3]^T, \vec{v} = [5, 3, 2]^T$
- 9) $\vec{u} = [4, 5, 5]^T, \vec{v} = [3, 5, -1]^T$
- 10) $\vec{u} = [-5, 8, 4]^T, \vec{v} = [6, 3, 3]^T$
- 11) $\vec{u} = [-5, 6, 3]^T, \vec{v} = [7, -3, -1]^T$
- 12) $\vec{u} = [-8, 8, 5]^T, \vec{v} = [-5, 6, -8]^T$
- 13) $\vec{u} = [4, 2, 7]^T, \vec{v} = [6, -7, 8]^T$
- 14) $\vec{u} = [4, -5, -7]^T, \vec{v} = [2, 7, 9]^T$
- 15) $\vec{u} = [8, 5, -2]^T, \vec{v} = [-4, -7, 8]^T$
- 16) $\vec{u} = [-6, 6, -1]^T, \vec{v} = [-5, 8, 9]^T$
- 17) $\vec{u} = [3, 2, 4]^T, \vec{v} = [-5, 8, -2]^T$
- 18) $\vec{u} = [6, -1, -5]^T, \vec{v} = [-8, 3, 6]^T$
- 19) $\vec{u} = [9, -8, -8]^T, \vec{v} = [6, -7, 6]^T$
- 20) $\vec{u} = [4, -8, 4]^T, \vec{v} = [9, 2, 1]^T$

¹Translated,commented and edited by Kraut

²Date of the first issue of english translation : 24.09.2018

³"T" here stands for transposition. Simply put it means 'take a row and make a column out of it'. In this context it is used

to spare area in text layout and is equal to $\begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$

Solutions⁴:

$$1) \text{ a) } \vec{u} + \vec{v} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4+4 \\ 5+3 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ 3 \end{bmatrix}, \vec{u} - \vec{v} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4-4 \\ 5-3 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$b) |\vec{u}| = \sqrt{u_x^2 + u_y^2 + u_z^2} = \sqrt{16 + 25 + 4} = \sqrt{45} = 6.708, |\vec{v}| = \sqrt{26} = 5.099$$

$$c) \hat{u}_u = \frac{1}{|\vec{u}|} \vec{u} = \frac{1}{6.708} \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{6.708} \\ \frac{5}{6.708} \\ \frac{2}{6.708} \end{bmatrix} = \begin{bmatrix} 0.596 \\ 0.745 \\ 0.298 \end{bmatrix}, \hat{u}_v = \begin{bmatrix} 0.784 \\ 0.588 \\ 0.196 \end{bmatrix}$$

$$d) |\vec{u} - \vec{v}| = \sqrt{0^2 + 2^2 + 1^2} = \sqrt{5} = 2.236$$

$$e) \vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z = 16 + 15 + 2 = 33$$

$$f) \cos \alpha = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} = \frac{33}{6.708 \cdot 5.099} = 0.965 \Rightarrow \alpha = \arccos(0.965) = 15.25^\circ$$

$$g) \vec{w} = \vec{u} \times \vec{v} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \times \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} u_y v_z - v_y u_z \\ u_z v_x - v_z u_x \\ u_x v_y - v_x u_y \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 - 3 \cdot 2 \\ 2 \cdot 4 - 1 \cdot 4 \\ 4 \cdot 3 - 4 \cdot 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -8 \end{bmatrix}$$

To check whether the answer is right an easy method is to try and calculate dot product between \vec{w} and \vec{u} or \vec{v} . Since \vec{w} is orthogonal to the other two, the respective dot products should equal 0. Let's try it out: $\vec{w} \cdot \vec{v} = w_x v_x + w_y v_y + w_z v_z = -1 \cdot 4 + 4 \cdot 3 + (-8) \cdot 1 = -4 + 12 - 8 = 0$. (Correct!)

h) To describe the positive z -axis we can use the unit vector $\hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. To calculate angle between \hat{z} and \vec{w}

we again use dot product: $\cos \beta = \frac{\vec{w} \cdot \hat{z}}{|\vec{w}| |\hat{z}|} = \frac{-8}{9 \cdot 1} = -0.888 \Rightarrow \beta = \arccos(-0.888) = 152.73^\circ$

$$i) \vec{l}(\lambda) = \vec{u} + \lambda(\vec{v} - \vec{u}) = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$$

j) Since $x-y$ plane is horizontal and it's z -coordinate is 0 at all places we can use this coordinate in our line equation $\vec{l}(\lambda)$. We use z -components of our line equation. When they equal 0 the line is at the point P_0 ,

$$\text{where it pierces through the horizontal } x-y\text{-plane. } 2 - \lambda \cdot 1 = 0 \Rightarrow \lambda = 2 \Rightarrow P_0 = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} = {}^5 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

k) First we normalize the direction vector, which we name $\vec{d} = \vec{v} - \vec{u}$ so that if we multiply it's unit vector by any integer value it moves us by that many length units in the particular direction.

$$\hat{d} = \frac{1}{|\vec{d}|} \vec{d} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.894 \\ -0.447 \end{bmatrix}. \text{ Now we can insert the distance } +17 \text{ or } -17 \text{ as } \lambda \text{ and use } \hat{d} \text{ in our line}$$

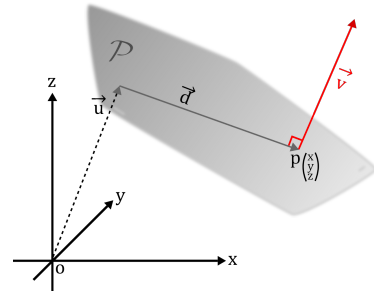
equation to calculate points P_1 and P_2 :

$$P_1 = \vec{u} + 17\hat{d} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 17 \cdot (-0.894) \\ 17 \cdot (-0.447) \end{bmatrix} = \begin{bmatrix} 4 \\ -10.20 \\ -5.60 \end{bmatrix}, P_2 = \vec{u} - 17\hat{d} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -17 \cdot (-0.894) \\ -17 \cdot (-0.447) \end{bmatrix} = \begin{bmatrix} 4 \\ 20.20 \\ 9.60 \end{bmatrix}$$

l) To calculate standard equation from point-normal form we choose a random point $p = (x, y, z)^T$ on the plane and use a vector \vec{d} (which is parallel to or lies within the plane \mathcal{P}) so that $\vec{p} = \vec{u} + \vec{d}$. We apply dot-product with \vec{v} on both sides of the equation: $\vec{p} \cdot \vec{v} = \vec{u} \cdot \vec{v} + \vec{d} \cdot \vec{v}$. Since $\vec{d} \perp \vec{v}$, their dot product equals zero. Therefore $\vec{p} \cdot \vec{v} = \vec{u} \cdot \vec{v}$.

$$\text{Now we can insert variables and numbers: } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \Rightarrow$$

$$4x + 3y + z = 33$$



⁴Translators note: I wrote out complete solution only for the first pair of vectors; the rest of answers are written in short form as in the original text

⁵in strict terms this equals sign is incorrect because right side of the equation contains point coordinates and left side calculates a vector to this point.

m) First let us find a formula to calculate distance between any given point b and a plane given in point-normal form with normal vector \vec{n} and point a in it. We also use vector \vec{c} that runs inside this plane or is parallel to it.

Line connecting plane \mathcal{P} and point b will be shortest if this line is orthogonal to the plane. We use vector \vec{d} that runs from the nearest point on the plane to point b . $\vec{b} = \vec{a} + \vec{c} + \vec{d}$. We apply dot product with \vec{n} on both sides of the equation $\Rightarrow \vec{b} \cdot \vec{n} = \vec{a} \cdot \vec{n} + \underbrace{\vec{c} \cdot \vec{n}}_0 + \vec{d} \cdot \vec{n}$. Since $\vec{d} \parallel \vec{n}$ we can express \vec{d} as \vec{n}

multiplied by a scalar $\lambda \Rightarrow \vec{d} = \lambda \vec{n}$. We insert this in previous equation $\Rightarrow \vec{b} \cdot \vec{n} = \vec{a} \cdot \vec{n} + (\lambda \vec{n}) \cdot \vec{n}$. From this we can calculate value of λ and therefore length of \vec{d} . \Rightarrow

$$\lambda = \frac{\vec{b} \cdot \vec{n} - \vec{a} \cdot \vec{n}}{\vec{n} \cdot \vec{n}}. \text{ Distance } \delta_P = |\vec{d}| = |\lambda \vec{n}| = |\lambda| |\vec{n}| = \left| \frac{\vec{b} \cdot \vec{n} - \vec{a} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \right| |\vec{n}|, \text{ therefore } \delta_P = \frac{|\vec{b} \cdot \vec{n} - \vec{a} \cdot \vec{n}|}{|\vec{n}|}. \text{ In}$$

our case point b is the origin ($\vec{0}$) with coordinates $(0,0,0)^T$, \vec{a} is \vec{u} and \vec{n} is \vec{v} . Let's insert the values:

$$\delta_P = \frac{\underbrace{|\vec{0} \cdot \vec{v} - \vec{u} \cdot \vec{v}|}_0}{|\vec{v}|} = \frac{|-33|}{|\sqrt{26}|} = 6.47$$

n) y -axis x - and z -coordinates are 0 at all points. So we set x and z to 0 in plane equation and can then calculate y -coordinate of point P_I : $4 \cdot 0 + 3y + 0 = 33 \Rightarrow y = 33/3 = 11 \Rightarrow P_I = (0, 11, 0)^T$

o) $x - y$ -plane's z -coordinate is 0 at all it's points. Accordingly we set $z = 0$ in our plane equation: $4x + 3y = 33 \Rightarrow 3y = -4x + 33 \Rightarrow y = -\frac{4}{3}x + 11$

$$\begin{aligned} 2) \vec{u} + \vec{v} &= [6, 4, 3]^T, \vec{u} - \vec{v} = [2, -12, 1]^T, |\vec{u}| = \sqrt{36} = 6.000, |\vec{v}| = \sqrt{69} = 8.307 \\ \hat{u}_u &= [0.667, -0.667, 0.333]^T, \hat{u}_v = [0.241, 0.963, 0.120]^T, |\vec{u} - \vec{v}| = \sqrt{149} = 12.207, \vec{u} \cdot \vec{v} = -22 \\ \cos \alpha &= -22/\sqrt{36 \cdot 69} = -0.441, \alpha = 2.028 = 116.19^\circ, \vec{u} \times \vec{v} = [-20, 0, 40]^T \\ \cos \beta &= 40/\sqrt{2000} = 0.894, \beta = 0.464 = 26.57^\circ \\ l: \vec{l}(\lambda) &= [4, -4, 2]^T + \lambda[-2, 12, -1]^T \text{ or: } \vec{l}(\mu) = [2, 8, 1]^T + \mu[2, -12, 1]^T \\ P_0 : 2 + -1\lambda &= 0 \Rightarrow \lambda = 2.0000 \Rightarrow P_0 = (0.00, 20.00, 0)^T \\ P_1, P_2 : 17/|\vec{v} - \vec{u}| &= 17/12.21 = 1.39^6 \\ \Rightarrow P_1 &= \vec{u} + 1.39(\vec{v} - \vec{u}) = (1.21, 12.71, 0.61)^T, P_2 = \vec{u} - 1.39(\vec{v} - \vec{u}) = (6.79, -20.71, 3.39)^T \\ \mathcal{P} : 2x + 8y + z &= -22, \delta_P = |-22|/|\vec{v}| = 22/8.31 = 2.65 \\ P_I : 8y &= -22 \Rightarrow P_I = (0, -2.75, 0)^T, l_I : 2x + 8y = -22 \Rightarrow y = -0.25x - 2.75 \end{aligned}$$

$$\begin{aligned} 3) \vec{u} + \vec{v} &= [14, -1, -3]^T, \vec{u} - \vec{v} = [0, -7, 13]^T, |\vec{u}| = \sqrt{90} = 9.487, |\vec{v}| = \sqrt{122} = 11.045 \\ \hat{u}_u &= [0.738, -0.422, 0.527]^T, \hat{u}_v = [0.634, 0.272, -0.724]^T, |\vec{u} - \vec{v}| = \sqrt{218} = 14.765, \vec{u} \cdot \vec{v} = -3 \\ \cos \alpha &= -3/\sqrt{90 \cdot 122} = -0.029, \alpha = 1.599 = 91.64^\circ, \vec{u} \times \vec{v} = [17, 91, 49]^T \\ \cos \beta &= 49/\sqrt{10971} = 0.468, \beta = 1.084 = 62.11^\circ \\ l: \vec{l}(\lambda) &= [7, -4, 5]^T + \lambda[0, 7, -13]^T \text{ or: } \vec{l}(\mu) = [7, 3, -8]^T + \mu[0, -7, 13]^T \\ P_0 : 5 + -13\lambda &= 0 \Rightarrow \lambda = 0.3846 \Rightarrow P_0 = (7.00, -1.31, 0)^T \\ P_1, P_2 : 17/|\vec{v} - \vec{u}| &= 17/14.76 = 1.15 \\ \Rightarrow P_1 &= \vec{u} + 1.15(\vec{v} - \vec{u}) = (7.00, 4.06, -9.97)^T, P_2 = \vec{u} - 1.15(\vec{v} - \vec{u}) = (7.00, -12.06, 19.97)^T \\ \mathcal{P} : 7x + 3y - 8z &= -3, \delta_P = |-3|/|\vec{v}| = 3/11.05 = 0.27 \\ P_I : 3y &= -3 \Rightarrow P_I = (0, -1.00, 0)^T, l_I : 7x + 3y = -3 \Rightarrow y = -2.33x - 1.00 \end{aligned}$$

$$\begin{aligned} 4) \vec{u} + \vec{v} &= [13, 4, 3]^T, \vec{u} - \vec{v} = [-3, 6, 1]^T, |\vec{u}| = \sqrt{54} = 7.348, |\vec{v}| = \sqrt{66} = 8.124 \\ \hat{u}_u &= [0.680, 0.680, 0.272]^T, \hat{u}_v = [0.985, -0.123, 0.123]^T, |\vec{u} - \vec{v}| = \sqrt{46} = 6.782, \vec{u} \cdot \vec{v} = 37 \\ \cos \alpha &= 37/\sqrt{54 \cdot 66} = 0.620, \alpha = 0.902 = 51.70^\circ, \vec{u} \times \vec{v} = [7, 11, -45]^T \\ \cos \beta &= -45/\sqrt{2195} = -0.960, \beta = 2.860 = 163.84^\circ \\ l: \vec{l}(\lambda) &= [5, 5, 2]^T + \lambda[3, -6, -1]^T \text{ or: } \vec{l}(\mu) = [8, -1, 1]^T + \mu[-3, 6, 1]^T \end{aligned}$$

⁶W.Rosenheinrich uses another approach to solve the question than the one in solution provided by the translator. He first calculates the ratio between 17 and length of the vector, so that later he can multiply vector with this ratio and therefore it's length becomes 17

$$\begin{aligned}
P_0 : 2 + -1\lambda = 0 &\Rightarrow \lambda = 2.0000 \Rightarrow P_0 = (11.00, -7.00, 0)^T \\
P_1, P_2 : 17/|\vec{v} - \vec{u}| &= 17/6.78 = 2.51 \\
\Rightarrow P_1 = \vec{u} + 2.51(\vec{v} - \vec{u}) &= (12.52, -10.04, -0.51)^T, P_2 = \vec{u} - 2.51(\vec{v} - \vec{u}) = (-2.52, 20.04, 4.51)^T \\
\mathcal{P} : 8x - y + z = 37, \delta_P &= |37|/|\vec{v}| = 37/8.12 = 4.55 \\
P_I : -1y = 37 &\Rightarrow P_I = (0, -37.00, 0)^T, l_I : 8x - y = 37 \Rightarrow y = 8.00x - 37.00
\end{aligned}$$

$$\begin{aligned}
5) \vec{u} + \vec{v} &= [-14, 4, -5]^T, \vec{u} - \vec{v} = [4, 12, 9]^T, |\vec{u}| = \sqrt{93} = 9.644, |\vec{v}| = \sqrt{146} = 12.083 \\
\hat{u}_u &= [-0.518, 0.830, 0.207]^T, \hat{u}_v = [-0.745, -0.331, -0.579]^T, |\vec{u} - \vec{v}| = \sqrt{241} = 15.524, \vec{u} \cdot \vec{v} = -1 \\
\cos \alpha &= -1/\sqrt{93 \cdot 146} = -0.009, \alpha = 1.579 = 90.49^\circ, \vec{u} \times \vec{v} = [-48, -53, 92]^T \\
\cos \beta &= 92/\sqrt{13577} = 0.790, \beta = 0.661 = 37.86^\circ \\
l: \vec{l}(\lambda) &= [-5, 8, 2]^T + \lambda[-4, -12, -9]^T \text{ or } l: \vec{l}(\mu) = [-9, -4, -7]^T + \mu[4, 12, 9]^T \\
P_0 : 2 + -9\lambda = 0 &\Rightarrow \lambda = 0.2222 \Rightarrow P_0 = (-5.89, 5.33, 0)^T \\
P_1, P_2 : 17/|\vec{v} - \vec{u}| &= 17/15.52 = 1.10 \\
\Rightarrow P_1 = \vec{u} + 1.10(\vec{v} - \vec{u}) &= (-9.38, -5.14, -7.86)^T, P_2 = \vec{u} - 1.10(\vec{v} - \vec{u}) = (-0.62, 21.14, 11.86)^T \\
\mathcal{P} : -9x - 4y - 7z &= -1, \delta_P = |-1|/|\vec{v}| = 1/12.08 = 0.08 \\
P_I : -4y = -1 &\Rightarrow P_I = (0, 0.25, 0)^T, l_I : -9x - 4y = -1 \Rightarrow y = -2.25x + 0.25
\end{aligned}$$

$$\begin{aligned}
6) \vec{u} + \vec{v} &= [4, -14, 2]^T, \vec{u} - \vec{v} = [0, -2, -14]^T, |\vec{u}| = \sqrt{104} = 10.198, |\vec{v}| = \sqrt{104} = 10.198 \\
\hat{u}_u &= [0.196, -0.784, -0.588]^T, \hat{u}_v = [0.196, -0.588, 0.784]^T, |\vec{u} - \vec{v}| = \sqrt{200} = 14.142, \vec{u} \cdot \vec{v} = 4 \\
\cos \alpha &= 4/\sqrt{104 \cdot 104} = 0.038, \alpha = 1.532 = 87.80^\circ, \vec{u} \times \vec{v} = [-100, -28, 4]^T \\
\cos \beta &= 4/\sqrt{10800} = 0.038, \beta = 1.532 = 87.79^\circ \\
l: \vec{l}(\lambda) &= [2, -8, -6]^T + \lambda[0, 2, 14]^T \text{ or } l: \vec{l}(\mu) = [2, -6, 8]^T + \mu[0, -2, -14]^T \\
P_0 : -6 + 14\lambda = 0 &\Rightarrow \lambda = 0.4286 \Rightarrow P_0 = (2.00, -7.14, 0)^T \\
P_1, P_2 : 17/|\vec{v} - \vec{u}| &= 17/14.14 = 1.20 \\
\Rightarrow P_1 = \vec{u} + 1.20(\vec{v} - \vec{u}) &= (2.00, -5.60, 10.83)^T, P_2 = \vec{u} - 1.20(\vec{v} - \vec{u}) = (2.00, -10.40, -22.83)^T \\
\mathcal{P} : 2x - 6y + 8z &= 4, \delta_P = |4|/|\vec{v}| = 4/10.20 = 0.39 \\
P_I : -6y = 4 &\Rightarrow P_I = (0, -0.67, 0)^T, l_I : 2x - 6y = 4 \Rightarrow y = 0.33x - 0.67
\end{aligned}$$

$$\begin{aligned}
7) \vec{u} + \vec{v} &= [7, 4, 0]^T, \vec{u} - \vec{v} = [-1, 6, 14]^T, |\vec{u}| = \sqrt{83} = 9.110, |\vec{v}| = \sqrt{66} = 8.124 \\
\hat{u}_u &= [0.329, 0.549, 0.768]^T, \hat{u}_v = [0.492, -0.123, -0.862]^T, |\vec{u} - \vec{v}| = \sqrt{233} = 15.264, \vec{u} \cdot \vec{v} = -42 \\
\cos \alpha &= -42/\sqrt{83 \cdot 66} = -0.567, \alpha = 2.174 = 124.57^\circ, \vec{u} \times \vec{v} = [-28, 49, -23]^T \\
\cos \beta &= -23/\sqrt{3714} = -0.377, \beta = 1.958 = 112.17^\circ \\
l: \vec{l}(\lambda) &= [3, 5, 7]^T + \lambda[1, -6, -14]^T \text{ or } l: \vec{l}(\mu) = [4, -1, -7]^T + \mu[-1, 6, 14]^T \\
P_0 : 7 + -14\lambda = 0 &\Rightarrow \lambda = 0.5000 \Rightarrow P_0 = (3.50, 2.00, 0)^T \\
P_1, P_2 : 17/|\vec{v} - \vec{u}| &= 17/15.26 = 1.11 \\
\Rightarrow P_1 = \vec{u} + 1.11(\vec{v} - \vec{u}) &= (4.11, -1.68, -8.59)^T, P_2 = \vec{u} - 1.11(\vec{v} - \vec{u}) = (1.89, 11.68, 22.59)^T \\
\mathcal{P} : 4x - y - 7z &= -42, \delta_P = |-42|/|\vec{v}| = 42/8.12 = 5.17 \\
P_I : -1y = -42 &\Rightarrow P_I = (0, 42.00, 0)^T, l_I : 4x - y = -42 \Rightarrow y = 4.00x + 42.00
\end{aligned}$$

$$\begin{aligned}
8) \vec{u} + \vec{v} &= [9, 7, 5]^T, \vec{u} - \vec{v} = [-1, 1, 1]^T, |\vec{u}| = \sqrt{41} = 6.403, |\vec{v}| = \sqrt{38} = 6.164 \\
\hat{u}_u &= [0.625, 0.625, 0.469]^T, \hat{u}_v = [0.811, 0.487, 0.324]^T, |\vec{u} - \vec{v}| = \sqrt{3} = 1.732, \vec{u} \cdot \vec{v} = 38 \\
\cos \alpha &= 38/\sqrt{41 \cdot 38} = 0.963, \alpha = 0.274 = 15.69^\circ, \vec{u} \times \vec{v} = [-1, 7, -8]^T \\
\cos \beta &= -8/\sqrt{114} = -0.749, \beta = 2.418 = 138.53^\circ \\
l: \vec{l}(\lambda) &= [4, 4, 3]^T + \lambda[1, -1, -1]^T \text{ or } l: \vec{l}(\mu) = [5, 3, 2]^T + \mu[-1, 1, 1]^T \\
P_0 : 3 + -1\lambda = 0 &\Rightarrow \lambda = 3.0000 \Rightarrow P_0 = (7.00, 1.00, 0)^T \\
P_1, P_2 : 17/|\vec{v} - \vec{u}| &= 17/1.73 = 9.81 \\
\Rightarrow P_1 = \vec{u} + 9.81(\vec{v} - \vec{u}) &= (13.81, -5.81, -6.81)^T, P_2 = \vec{u} - 9.81(\vec{v} - \vec{u}) = (-5.81, 13.81, 12.81)^T \\
\mathcal{P} : 5x + 3y + 2z &= 38, \delta_P = |38|/|\vec{v}| = 38/6.16 = 6.16 \\
P_I : 3y = 38 &\Rightarrow P_I = (0, 12.67, 0)^T, l_I : 5x + 3y = 38 \Rightarrow y = -1.67x + 12.67
\end{aligned}$$

$$\begin{aligned}
9) \vec{u} + \vec{v} &= [7, 10, 4]^T, \vec{u} - \vec{v} = [1, 0, 6]^T, |\vec{u}| = \sqrt{66} = 8.124, |\vec{v}| = \sqrt{35} = 5.916 \\
\hat{u}_u &= [0.492, 0.615, 0.615]^T, \hat{u}_v = [0.507, 0.845, -0.169]^T, |\vec{u} - \vec{v}| = \sqrt{37} = 6.083, \vec{u} \cdot \vec{v} = 32
\end{aligned}$$

$$\cos \alpha = 32/\sqrt{66 \cdot 35} = 0.666, \alpha = 0.842 = 48.26^\circ, \vec{u} \times \vec{v} = [-30, 19, 5]^T$$

$$\cos \beta = 5/\sqrt{1286} = 0.139, \beta = 1.431 = 81.99^\circ$$

$$l: \vec{l}(\lambda) = [4, 5, 5]^T + \lambda[-1, 0, -6]^T \text{ or: } \vec{l}(\mu) = [3, 5, -1]^T + \mu[1, 0, 6]^T$$

$$P_0: 5 + -6\lambda = 0 \Rightarrow \lambda = 0.8333 \Rightarrow P_0 = (3.17, 5.00, 0)^T$$

$$P_1, P_2: 17/|\vec{v} - \vec{u}| = 17/6.08 = 2.79$$

$$\Rightarrow P_1 = \vec{u} + 2.79(\vec{v} - \vec{u}) = (1.21, 5.00, -11.77)^T, P_2 = \vec{u} - 2.79(\vec{v} - \vec{u}) = (6.79, 5.00, 21.77)^T$$

$$\mathcal{P}: 3x + 5y - z = 32, \delta_P = |32|/|\vec{v}| = 32/5.92 = 5.41$$

$$P_I: 5y = 32 \Rightarrow P_I = (0, 6.40, 0)^T, l_I: 3x + 5y = 32 \Rightarrow y = -0.60x + 6.40$$

$$10) \vec{u} + \vec{v} = [1, 11, 7]^T, \vec{u} - \vec{v} = [-11, 5, 1]^T, |\vec{u}| = \sqrt{105} = 10.247, |\vec{v}| = \sqrt{54} = 7.348$$

$$\hat{u}_u = [-0.488, 0.781, 0.390]^T, \hat{u}_v = [0.816, 0.408, 0.408]^T, |\vec{u} - \vec{v}| = \sqrt{147} = 12.124, \vec{u} \cdot \vec{v} = 6$$

$$\cos \alpha = 6/\sqrt{105 \cdot 54} = 0.080, \alpha = 1.491 = 85.43^\circ, \vec{u} \times \vec{v} = [12, 39, -63]^T$$

$$\cos \beta = -63/\sqrt{5634} = -0.839, \beta = 2.567 = 147.07^\circ$$

$$l: \vec{l}(\lambda) = [-5, 8, 4]^T + \lambda[11, -5, -1]^T \text{ or: } \vec{l}(\mu) = [6, 3, 3]^T + \mu[-11, 5, 1]^T$$

$$P_0: 4 + -1\lambda = 0 \Rightarrow \lambda = 4.0000 \Rightarrow P_0 = (39.00, -12.00, 0)^T$$

$$P_1, P_2: 17/|\vec{v} - \vec{u}| = 17/12.12 = 1.40$$

$$\Rightarrow P_1 = \vec{u} + 1.40(\vec{v} - \vec{u}) = (10.42, 0.99, 2.60)^T, P_2 = \vec{u} - 1.40(\vec{v} - \vec{u}) = (-20.42, 15.01, 5.40)^T$$

$$\mathcal{P}: 6x + 3y + 3z = 6, \delta_P = |6|/|\vec{v}| = 6/7.35 = 0.82 \quad P_I: 3y = 6 \Rightarrow P_I = (0, 2.00, 0)^T, l_I: 6x + 3y = 6 \Rightarrow y = -2.00x + 2.00$$

$$11) \vec{u} + \vec{v} = [2, 3, 2]^T, \vec{u} - \vec{v} = [-12, 9, 4]^T, |\vec{u}| = \sqrt{70} = 8.367, |\vec{v}| = \sqrt{59} = 7.681$$

$$\hat{u}_u = [-0.598, 0.717, 0.359]^T, \hat{u}_v = [0.911, -0.391, -0.130]^T, |\vec{u} - \vec{v}| = \sqrt{241} = 15.524, \vec{u} \cdot \vec{v} = -56$$

$$\cos \alpha = -56/\sqrt{70 \cdot 59} = -0.871, \alpha = 2.629 = 150.62^\circ, \vec{u} \times \vec{v} = [3, 16, -27]^T$$

$$\cos \beta = -27/\sqrt{994} = -0.856, \beta = 2.599 = 148.91^\circ$$

$$l: \vec{l}(\lambda) = [-5, 6, 3]^T + \lambda[12, -9, -4]^T \text{ or: } \vec{l}(\mu) = [7, -3, -1]^T + \mu[-12, 9, 4]^T$$

$$P_0: 3 + -4\lambda = 0 \Rightarrow \lambda = 0.7500 \Rightarrow P_0 = (4.00, -0.75, 0)^T$$

$$P_1, P_2: 17/|\vec{v} - \vec{u}| = 17/15.52 = 1.10$$

$$\Rightarrow P_1 = \vec{u} + 1.10(\vec{v} - \vec{u}) = (8.14, -3.86, -1.38)^T, P_2 = \vec{u} - 1.10(\vec{v} - \vec{u}) = (-18.14, 15.86, 7.38)^T$$

$$\mathcal{P}: 7x - 3y - z = -56, \delta_P = |-56|/|\vec{v}| = 56/7.68 = 7.29$$

$$P_I: -3y = -56 \Rightarrow P_I = (0, 18.67, 0)^T, l_I: 7x - 3y = -56 \Rightarrow y = 2.33x + 18.67$$

$$12) \vec{u} + \vec{v} = [-13, 14, -3]^T, \vec{u} - \vec{v} = [-3, 2, 13]^T, |\vec{u}| = \sqrt{153} = 12.369, |\vec{v}| = \sqrt{125} = 11.180$$

$$\hat{u}_u = [-0.647, 0.647, 0.404]^T, \hat{u}_v = [-0.447, 0.537, -0.716]^T, |\vec{u} - \vec{v}| = \sqrt{182} = 13.491, \vec{u} \cdot \vec{v} = 48$$

$$\cos \alpha = 48/\sqrt{153 \cdot 125} = 0.347, \alpha = 1.216 = 69.69^\circ, \vec{u} \times \vec{v} = [-94, -89, -8]^T$$

$$\cos \beta = -8/\sqrt{16821} = -0.062, \beta = 1.633 = 93.54^\circ$$

$$l: \vec{l}(\lambda) = [-8, 8, 5]^T + \lambda[3, -2, -13]^T \text{ or } l: \vec{l}(\mu) = [-5, 6, -8]^T + \mu[-3, 2, 13]^T$$

$$P_0: 5 + -13\lambda = 0 \Rightarrow \lambda = 0.3846 \Rightarrow P_0 = (-6.85, 7.23, 0)^T$$

$$P_1, P_2: 17/|\vec{v} - \vec{u}| = 17/13.49 = 1.26$$

$$\Rightarrow P_1 = \vec{u} + 1.26(\vec{v} - \vec{u}) = (-4.22, 5.48, -11.38)^T, P_2 = \vec{u} - 1.26(\vec{v} - \vec{u}) = (-11.78, 10.52, 21.38)^T$$

$$\mathcal{P}: -5x + 6y - 8z = 48, \delta_P = |48|/|\vec{v}| = 48/11.18 = 4.29$$

$$P_I: 6y = 48 \Rightarrow P_I = (0, 8.00, 0)^T, l_I: -5x + 6y = 48 \Rightarrow y = 0.83x + 8.00$$

$$13) \vec{u} + \vec{v} = [10, -5, 15]^T, \vec{u} - \vec{v} = [-2, 9, -1]^T, |\vec{u}| = \sqrt{69} = 8.307, |\vec{v}| = \sqrt{149} = 12.207$$

$$\hat{u}_u = [0.482, 0.241, 0.843]^T, \hat{u}_v = [0.492, -0.573, 0.655]^T, |\vec{u} - \vec{v}| = \sqrt{86} = 9.274, \vec{u} \cdot \vec{v} = 66$$

$$\cos \alpha = 66/\sqrt{69 \cdot 149} = 0.651, \alpha = 0.862 = 49.39^\circ, \vec{u} \times \vec{v} = [65, 10, -40]^T$$

$$\cos \beta = -40/\sqrt{5925} = -0.520, \beta = 2.117 = 121.31^\circ$$

$$l: \vec{l}(\lambda) = [4, 2, 7]^T + \lambda[2, -9, 1]^T \text{ or } l: \vec{l}(\mu) = [6, -7, 8]^T + \mu[-2, 9, -1]^T$$

$$P_0: 7 + 1\lambda = 0 \Rightarrow \lambda = -7.0000 \Rightarrow P_0 = (-10.00, 65.00, 0)^T \quad P_1, P_2: 17/|\vec{v} - \vec{u}| = 17/9.27 = 1.83$$

$$\Rightarrow P_1 = \vec{u} + 1.83(\vec{v} - \vec{u}) = (7.67, -14.50, 8.83)^T, P_2 = \vec{u} - 1.83(\vec{v} - \vec{u}) = (0.33, 18.50, 5.17)^T$$

$$\mathcal{P}: 6x - 7y + 8z = 66, \delta_P = |66|/|\vec{v}| = 66/12.21 = 5.41 \quad P_I: -7y = 66 \Rightarrow P_I = (0, -9.43, 0)^T, l_I:$$

$$6x - 7y = 66 \Rightarrow y = 0.86x - 9.43$$

14) $\vec{u} + \vec{v} = [6, 2, 2]^T$, $\vec{u} - \vec{v} = [2, -12, -16]^T$, $|\vec{u}| = \sqrt{90} = 9.487$, $|\vec{v}| = \sqrt{134} = 11.576$
 $\hat{u}_u = [0.422, -0.527, -0.738]^T$, $\hat{u}_v = [0.173, 0.605, 0.777]^T$, $|\vec{u} - \vec{v}| = \sqrt{404} = 20.100$, $\vec{u} \cdot \vec{v} = -90$
 $\cos \alpha = -90/\sqrt{90 \cdot 134} = -0.820$, $\alpha = 2.531 = 145.04^\circ$, $\vec{u} \times \vec{v} = [4, -50, 38]^T$
 $\cos \beta = 38/\sqrt{3960} = 0.604$, $\beta = 0.922 = 52.85^\circ$
 $l: \vec{l}(\lambda) = [4, -5, -7]^T + \lambda[-2, 12, 16]^T$ or: $\vec{l}(\mu) = [2, 7, 9]^T + \mu[2, -12, -16]^T$
 $P_0: -7 + 16\lambda = 0 \Rightarrow \lambda = 0.4375 \Rightarrow P_0 = (3.13, 0.25, 0)^T$
 $P_1, P_2: 17/|\vec{v} - \vec{u}| = 17/20.10 = 0.85$
 $\Rightarrow P_1 = \vec{u} + 0.85(\vec{v} - \vec{u}) = (2.31, 5.15, 6.53)^T$, $P_2 = \vec{u} - 0.85(\vec{v} - \vec{u}) = (5.69, -15.15, -20.53)^T$
 $\mathcal{P}: 2x + 7y + 9z = -90$, $\delta_P = |-90|/|\vec{v}| = 90/11.58 = 7.77$
 $P_I: 7y = -90 \Rightarrow P_I = (0, -12.86, 0)^T$, $l_I: 2x + 7y = -90 \Rightarrow y = -0.29x - 12.86$

15) $\vec{u} + \vec{v} = [4, -2, 6]^T$, $\vec{u} - \vec{v} = [12, 12, -10]^T$, $|\vec{u}| = \sqrt{93} = 9.644$, $|\vec{v}| = \sqrt{129} = 11.358$
 $\hat{u}_u = [0.830, 0.518, -0.207]^T$, $\hat{u}_v = [-0.352, -0.616, 0.704]^T$, $|\vec{u} - \vec{v}| = \sqrt{388} = 19.698$, $\vec{u} \cdot \vec{v} = -83$
 $\cos \alpha = -83/\sqrt{93 \cdot 129} = -0.758$, $\alpha = 2.431 = 139.27^\circ$, $\vec{u} \times \vec{v} = [26, -56, -36]^T$
 $\cos \beta = -36/\sqrt{5108} = -0.504$, $\beta = 2.099 = 120.25^\circ$
 $l: \vec{l}(\lambda) = [8, 5, -2]^T + \lambda[-12, -12, 10]^T$ or $l: \vec{l}(\mu) = [-4, -7, 8]^T + \mu[12, 12, -10]^T$
 $P_0: -2 + 10\lambda = 0 \Rightarrow \lambda = 0.2000 \Rightarrow P_0 = (5.60, 2.60, 0)^T$ $P_1, P_2: 17/|\vec{v} - \vec{u}| = 17/19.70 = 0.86$
 $\Rightarrow P_1 = \vec{u} + 0.86(\vec{v} - \vec{u}) = (-2.36, -5.36, 6.63)^T$, $P_2 = \vec{u} - 0.86(\vec{v} - \vec{u}) = (18.36, 15.36, -10.63)^T$
 $\mathcal{P}: -4x - 7y + 8z = -83$, $\delta_P = |-83|/|\vec{v}| = 83/11.36 = 7.31$
 $P_I: -7y = -83 \Rightarrow P_I = (0, 11.86, 0)^T$, $l_I: -4x - 7y = -83 \Rightarrow y = -0.57x + 11.86$

16) $\vec{u} + \vec{v} = [-11, 14, 8]^T$, $\vec{u} - \vec{v} = [-1, -2, -10]^T$, $|\vec{u}| = \sqrt{73} = 8.544$, $|\vec{v}| = \sqrt{170} = 13.038$
 $\hat{u}_u = [-0.702, 0.702, -0.117]^T$, $\hat{u}_v = [-0.383, 0.614, 0.690]^T$, $|\vec{u} - \vec{v}| = \sqrt{105} = 10.247$, $\vec{u} \cdot \vec{v} = 69$
 $\cos \alpha = 69/\sqrt{73 \cdot 170} = 0.619$, $\alpha = 0.903 = 51.73^\circ$, $\vec{u} \times \vec{v} = [62, 59, -18]^T$
 $\cos \beta = -18/\sqrt{7649} = -0.206$, $\beta = 1.778 = 101.88^\circ$
 $l: \vec{l}(\lambda) = [-6, 6, -1]^T + \lambda[1, 2, 10]^T$ or: $\vec{l}(\mu) = [-5, 8, 9]^T + \mu[-1, -2, -10]^T$
 $P_0: -1 + 10\lambda = 0 \Rightarrow \lambda = 0.1000 \Rightarrow P_0 = (-5.90, 6.20, 0)^T$
 $P_1, P_2: 17/|\vec{v} - \vec{u}| = 17/10.25 = 1.66$
 $\Rightarrow P_1 = \vec{u} + 1.66(\vec{v} - \vec{u}) = (-4.34, 9.32, 15.59)^T$, $P_2 = \vec{u} - 1.66(\vec{v} - \vec{u}) = (-7.66, 2.68, -17.59)^T$
 $\mathcal{P}: -5x + 8y + 9z = 69$, $\delta_P = |69|/|\vec{v}| = 69/13.04 = 5.29$
 $P_I: 8y = 69 \Rightarrow P_I = (0, 8.63, 0)^T$, $l_I: -5x + 8y = 69 \Rightarrow y = 0.63x + 8.63$

17) $\vec{u} + \vec{v} = [-2, 10, 2]^T$, $\vec{u} - \vec{v} = [8, -6, 6]^T$, $|\vec{u}| = \sqrt{29} = 5.385$, $|\vec{v}| = \sqrt{93} = 9.644$
 $\hat{u}_u = [0.557, 0.371, 0.743]^T$, $\hat{u}_v = [-0.518, 0.830, -0.207]^T$, $|\vec{u} - \vec{v}| = \sqrt{136} = 11.662$, $\vec{u} \cdot \vec{v} = -7$
 $\cos \alpha = -7/\sqrt{29 \cdot 93} = -0.135$, $\alpha = 1.706 = 97.75^\circ$, $\vec{u} \times \vec{v} = [-36, -14, 34]^T$
 $\cos \beta = 34/\sqrt{2648} = 0.661$, $\beta = 0.849 = 48.65^\circ$
 $l: \vec{l}(\lambda) = [3, 2, 4]^T + \lambda[-8, 6, -6]^T$ or: $\vec{l}(\mu) = [-5, 8, -2]^T + \mu[8, -6, 6]^T$
 $P_0: 4 + -6\lambda = 0 \Rightarrow \lambda = 0.6667 \Rightarrow P_0 = (-2.33, 6.00, 0)^T$
 $P_1, P_2: 17/|\vec{v} - \vec{u}| = 17/11.66 = 1.46$
 $\Rightarrow P_1 = \vec{u} + 1.46(\vec{v} - \vec{u}) = (-8.66, 10.75, -4.75)^T$, $P_2 = \vec{u} - 1.46(\vec{v} - \vec{u}) = (14.66, -6.75, 12.75)^T$
 $\mathcal{P}: -5x + 8y - 2z = -7$, $\delta_P = |-7|/|\vec{v}| = 7/9.64 = 0.73$
 $P_I: 8y = -7 \Rightarrow P_I = (0, -0.88, 0)^T$, $l_I: -5x + 8y = -7 \Rightarrow y = 0.63x - 0.88$

18) $\vec{u} + \vec{v} = [-2, 2, 1]^T$, $\vec{u} - \vec{v} = [14, -4, -11]^T$, $|\vec{u}| = \sqrt{62} = 7.874$, $|\vec{v}| = \sqrt{109} = 10.440$
 $\hat{u}_u = [0.762, -0.127, -0.635]^T$, $\hat{u}_v = [-0.766, 0.287, 0.575]^T$, $|\vec{u} - \vec{v}| = \sqrt{333} = 18.248$, $\vec{u} \cdot \vec{v} = -81$
 $\cos \alpha = -81/\sqrt{62 \cdot 109} = -0.985$, $\alpha = 2.970 = 170.17^\circ$, $\vec{u} \times \vec{v} = [9, 4, 10]^T$
 $\cos \beta = 10/\sqrt{197} = 0.712$, $\beta = 0.778 = 44.56^\circ$ $l: \vec{l}(\lambda) = [6, -1, -5]^T + \lambda[-14, 4, 11]^T$ or $l: \vec{l}(\mu) = [-8, 3, 6]^T + \mu[14, -4, -11]^T$
 $P_0: -5 + 11\lambda = 0 \Rightarrow \lambda = 0.4545 \Rightarrow P_0 = (-0.36, 0.82, 0)^T$ $P_1, P_2: 17/|\vec{v} - \vec{u}| = 17/18.25 = 0.93$
 $\Rightarrow P_1 = \vec{u} + 0.93(\vec{v} - \vec{u}) = (-7.04, 2.73, 5.25)^T$, $P_2 = \vec{u} - 0.93(\vec{v} - \vec{u}) = (19.04, -4.73, -15.25)^T$
 $\mathcal{P}: -8x + 3y + 6z = -81$, $\delta_P = |-81|/|\vec{v}| = 81/10.44 = 7.76$

$$P_I : 3y = -81 \Rightarrow P_I = (0, -27.00, 0)^T, l_I : -8x + 3y = -81 \Rightarrow y = 2.67x - 27.00$$

19) $\vec{u} + \vec{v} = [15, -15, -2]^T, \vec{u} - \vec{v} = [3, -1, -14]^T, |\vec{u}| = \sqrt{209} = 14.457, |\vec{v}| = \sqrt{121} = 11.000$
 $\hat{u}_u = [0.623, -0.553, -0.553]^T, \hat{u}_v = [0.545, -0.636, 0.545]^T, |\vec{u} - \vec{v}| = \sqrt{206} = 14.353, \vec{u} \cdot \vec{v} = 62$
 $\cos \alpha = 62/\sqrt{209 \cdot 121} = 0.390, \alpha = 1.170 = 67.05^\circ, \vec{u} \times \vec{v} = [-104, -102, -15]^T$
 $\cos \beta = -15/\sqrt{21445} = -0.102, \beta = 1.673 = 95.88^\circ$
 $l: \vec{l}(\lambda) = [9, -8, -8]^T + \lambda[-3, 1, 14]^T$ or $l: \vec{l}(\mu) = [6, -7, 6]^T + \mu[3, -1, -14]^T$
 $P_0 : -8 + 14\lambda = 0 \Rightarrow \lambda = 0.5714 \Rightarrow P_0 = (7.29, -7.43, 0)^T$
 $P_1, P_2 : 17/|\vec{v} - \vec{u}| = 17/14.35 = 1.18$
 $\Rightarrow P_1 = \vec{u} + 1.18(\vec{v} - \vec{u}) = (5.45, -6.82, 8.58)^T, P_2 = \vec{u} - 1.18(\vec{v} - \vec{u}) = (12.55, -9.18, -24.58)^T$
 $\mathcal{P} : 6x - 7y + 6z = 62, \delta_P = |62|/|\vec{v}| = 62/11.00 = 5.64$
 $P_I : -7y = 62 \Rightarrow P_I = (0, -8.86, 0)^T, l_I : 6x - 7y = 62 \Rightarrow y = 0.86x - 8.86$

20) $\vec{u} + \vec{v} = [13, -6, 5]^T, \vec{u} - \vec{v} = [-5, -10, 3]^T, |\vec{u}| = \sqrt{96} = 9.798, |\vec{v}| = \sqrt{86} = 9.274$
 $\hat{u}_u = [0.408, -0.816, 0.408]^T, \hat{u}_v = [0.970, 0.216, 0.108]^T, |\vec{u} - \vec{v}| = \sqrt{134} = 11.576, \vec{u} \cdot \vec{v} = 24$
 $\cos \alpha = 24/\sqrt{96 \cdot 86} = 0.264, \alpha = 1.303 = 74.68^\circ, \vec{u} \times \vec{v} = [-16, 32, 80]^T$
 $\cos \beta = 80/\sqrt{7680} = 0.913, \beta = 0.421 = 24.09^\circ$
 $l: \vec{l}(\lambda) = [4, -8, 4]^T + \lambda[5, 10, -3]^T$ or $l: \vec{l}(\mu) = [9, 2, 1]^T + \mu[-5, -10, 3]^T$
 $P_0 : 4 + -3\lambda = 0 \Rightarrow \lambda = 1.3333 \Rightarrow P_0 = (10.67, 5.33, 0)^T$
 $P_1, P_2 : 17/|\vec{v} - \vec{u}| = 17/11.58 = 1.47$
 $\Rightarrow P_1 = \vec{u} + 1.47(\vec{v} - \vec{u}) = (11.34, 6.69, -0.41)^T, P_2 = \vec{u} - 1.47(\vec{v} - \vec{u}) = (-3.34, -22.69, 8.41)^T$
 $\mathcal{P} : 9x + 2y + z = 24, \delta_P = |24|/|\vec{v}| = 24/9.27 = 2.59$
 $P_I : 2y = 24 \Rightarrow P_I = (0, 12.00, 0)^T, l_I : 9x + 2y = 24 \Rightarrow y = -4.50x + 12.00$

Control questions on vector geometry⁸

Questions:

1. What kind of conclusion can be drawn about a plane $ax + by + cz = d$,⁹ if *one* of the three terms on the left side of the equation is missing, as for example $ax + cz = d$?
What does it mean if *two* of the terms are missing, as in $by = d$?
2. Give a representation of a straight line that runs vertically through space (within the usual coordinate system) and passes through the point $(-2, 5, -2)^T$!¹⁰
3. What does it mean, when dot product of two normalized vectors \hat{n}_1 and \hat{n}_2 equals $+1$? What if it equals -1 ?
- 4.a) How to quickly and easily design a vector, that is orthogonal to $\vec{u} \in \mathbb{R}^2$?¹¹
b) How for $\vec{v} \in \mathbb{R}^3$?
c) What is the principal difference between those two tasks?
d) How to construct a vector, that is orthogonal to two vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$?
e) How in case if $\vec{u}, \vec{v} \in \mathbb{R}^3$?
5. What can be said about a dot product of two vectors with an acute angle between them?
6. There are two given line equations $l_1 : \vec{r}_{(1)}(\lambda) = \vec{a} + \lambda\vec{b}$ and $l_2 : \vec{r}_{(2)}(\mu) = \vec{c} + \mu\vec{d}$. How can we decide if these two lines are in fact *one and the same* ?
7. There are two given plane equations $\mathcal{P}_1 : a_1x + b_1y + c_1z = d_1$ and $\mathcal{P}_2 : a_2x + b_2y + c_2z = d_2$. How can we decide if these two planes are in fact *one and the same* ?
8. Two vectors with length l_1 and l_2 respectively are perpendicular to each other. What is the length of their sum?
9. Any two points $\vec{r}(\lambda_1)$ and $\vec{r}(\lambda_2)$ on the line $\vec{r}(\lambda) = \vec{a} + \lambda\vec{b}$ are in distance $|\lambda_2 - \lambda_1|$ from each other. What conclusion can be drawn here?
10. We got random line $\vec{l}(\lambda) = \vec{p} + \lambda\vec{q}$ and random plane $ax + by + cz = d$. How can we effectively prove that those two don't intersect?
11. A straight line in \mathbb{R}^2 is a picture of the linear function $y = mx + n$. Give us a normal vector to this line!
12. What can we conclude if $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$?
13. Give us the length of each vector in convenient figures !

$$\text{a) } [0, -291, 0]^T, \text{ b) } [0, -291, 291]^T, \text{ c) } [122, 244, 366]^T$$

14. Calculate length of the following vector in hand: $[2222, 1111, 2222]^T$!
15. Does $|\vec{a} + \vec{b}| = |\vec{a}|$, mean that \vec{b} is a null-vector?
16. How can one find the point \vec{r} , which observed from \vec{p} is at the same amount of distance from point \vec{q} ?
17. How can we describe vectors \vec{r} , whose end points are on the border of unit-circle?
18. A given force $\vec{F} = [f_1, f_2, f_3]^T$ is to be split up in 3 separate forces, that are parallel to coordinate axes. Submit those 3 forces!
19. Force-vector \vec{F}_1 is the sum of two forces \vec{F}_2 and \vec{F}_3 . Does this mean that $|\vec{F}_1| \geq |\vec{F}_2|$?

⁷Date of first issue of english translation: 01.10.2018

⁸Commentary related to the math course and exams this material was partially written for was deliberately excluded by editor

⁹This is the so called 'standard equation of a plane' which is dot product expansion of 'point-normal form of a plane'

¹⁰'T' here stands for transposition. Simply put it means 'take a row and make a column out of it'. In this context it is used to spare area in text layout and corresponds to $\begin{pmatrix} -2 \\ 5 \\ -2 \end{pmatrix}$

¹¹ \in means 'element of' and \mathbb{R}^2 means '2D' or plane.

20. \hat{e}_v is the unit vector of \vec{v} . With help of it describe the unit vectors of following vectors!

$$a) 2\vec{v} \quad b) 6.905 \cdot 10^{-13} \vec{v} \quad c) -251.62 \vec{v}$$

21. What can one conclude from $(\vec{v} + \vec{u}) \cdot \vec{w} = \vec{v} \cdot \vec{w}$?

22. What can we conclude about \vec{v} if $|\vec{v}| = v_z$?

23. Calculate distance between points $(7293, 9753)^T$ and $(7293, 6420)^T$ in your head!

24. Are there any vectors that are orthogonal to all vectors of their dimension?

25. Can one directly recognize if a plane goes through the origin if it is given in vector equation¹² form? What about point-normal form?

26. We have a given force-vector $\vec{f} = [6, -6, 14]^T N$.

a) Make a new force-vector \vec{f}_a out of \vec{f} that goes in the same direction but pulls 20% stronger!

b) Make a new force-vector \vec{f}_b that goes in the same direction but pulls 20 N stronger!

27. Submit(without using calculator) practical length values of following vectors:

$$\vec{a} = \begin{bmatrix} 17 \\ 34 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 209 \\ 0 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} 19 \\ 19 \\ -19 \end{bmatrix}$$

¹²Translation note: there are some conflicting opinions on naming of this on the internet and in literature. Some sources call it 'parametric equation'- others 'vector equation' of a plane. Anyhow, what is meant here is $\vec{P}(\lambda, \mu) = \vec{a} + \lambda\vec{v} + \mu\vec{w}$ with \vec{a} - a vector to any point in the plane; \vec{v} and \vec{w} - two non-parallel vectors in the given plane or parallel to it

Solutions:

1. The respective term doesn't play any role in the equation. If we have a point in this plane, then we can randomly change (in this case) y -coordinate and other two coordinates won't change because of that. If we calculate many points like that it creates a line parallel to the axis of missing variable. Which means that the plane is running parallel to the axis of the missing variable (in this case y -axis).

If there are two terms missing - it means that the plane is parallel to the two 'missing' axes, or orthogonal to the given (in this case y -axis) one. Vector $[0, b, 0]^T$ (where $b \neq 0$) is its normal vector.

2. 'Vertically' here means, that the x - and y -coordinates of other points on this line stay the same while z -coordinate changes. Whole set of points on this line can therefore be described by $(-2, 5, z)^T$.

Formal solution: direction vector of vertical line is vector $\vec{k} = [0, 0, 1]^T$, the point-direction form of the line

would therefore be $\vec{l} = \begin{bmatrix} -2 \\ 5 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ -2 + \lambda \end{bmatrix}$ where $-2 + \lambda = z$.

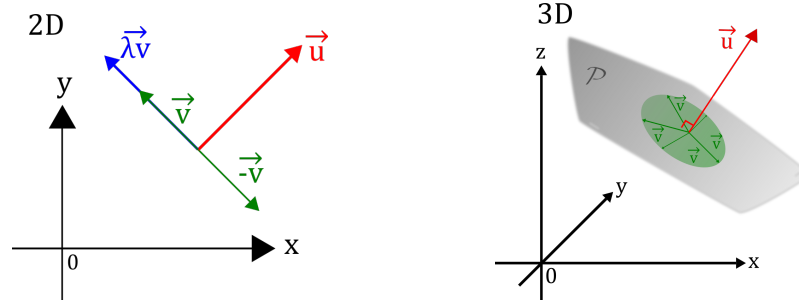
3. In first case (equals +1) it means that $\hat{n}_1 = \hat{n}_2$, in second case $\hat{n}_1 = -\hat{n}_2$.

4.

a) Exchange the places between components of \vec{u} and change one of their signs.

b) Set one of the components to 0, exchange the places of other two and change one of their signs.

c) In \mathbb{R}^2 the perpendicular vector is set except for its length and 2 directions. In \mathbb{R}^3 it's not the case - there is one degree of freedom left. The degree of freedom meant here is that of rotation in the plane orthogonal to \vec{u} . For three-dimensional object in space there are 3 degrees of freedom for rotation (around 3 of the axes) and 3 for translation (forward/backward, up/down, left/right).



d) It's generally not possible.

e) One makes a cross product of both.

5. It is positive.

6. Firstly: direction vectors must be parallel to each other (they do not necessarily need to point in the same direction though!), which means that one needs to be a multiple of the other: $\vec{b} = \xi \vec{d}$ or $\vec{d} = \eta \vec{b}$, with $\xi, \eta \neq 0$. Secondly: condition $\vec{a} \in l_2$ (or $\vec{c} \in l_1$ - it is the same) must be met, the system of 3 linear equations with one unknown factor μ must be solvable: $a_x = c_x + \mu d_x$, $a_y = c_y + \mu d_y$, $a_z = c_z + \mu d_z$. (this comes from each

variable line of: $\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} + \mu \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$)

7. One of the plane equations must be a multiple of other by a factor $\lambda \neq 0$.

8. $l = \sqrt{l_1^2 + l_2^2}$. Pythagoras.

9. \vec{b} is a unit vector: $|\vec{b}| = 1$

10. Dot product between normal vector $\vec{n} = (a, b, c)^T$ and \vec{q} must equal 0 $\Rightarrow \vec{n} \cdot \vec{q} = 0$ and $\vec{n} \cdot \vec{p} \neq d$.¹³ Alternatively: Insert line equation in equation of the plane. However it will result in the same calculations.

11. Standard form:¹⁴ $-mx + y = n$, normal vector is therefore $[-m, 1]^T$.

12. Those vectors are parallel to each other and point in the same direction. Borderline case: one of both is a null vector.

¹³if You are not able to understand why, see the solution of question 1) in "Practice material: vector math"

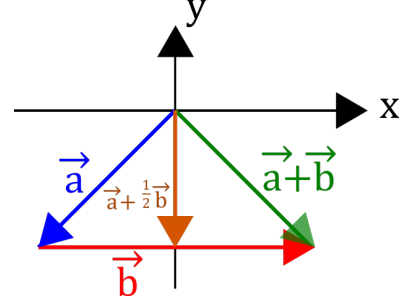
¹⁴analogue to the standard equation of a plane

13.

- a) $||[0, -291, 0]^T| = \sqrt{0^2 + (-291)^2 + 0^2} = 291,$
b) $||[0, -291, 291]^T| = |291| \cdot |[0, -1, 1]^T| = 291 \cdot \sqrt{0^2 + (-1)^2 + 1^2} = 291\sqrt{2},$
c) $||[122, 244, 366]^T| = 122 \cdot |[1, 2, 3]^T| = 122 \cdot \sqrt{1^2 + 2^2 + 3^2} = 122\sqrt{14}$

14. $||[2222, 1111, 2222]^T| = 1111|[2, 1, 2]^T| = 1111\sqrt{2^2 + 1^2 + 2^2} = 1111\sqrt{9} = 3333.$

15. No. It only means that \vec{a} and $\vec{a} + \vec{b}$ put up an isosceles triangle. Some more details: \vec{a} and $\vec{a} + \vec{b}$ are the legs of isosceles triangle with its corner at the origin. Angle bisector of the angle in this corner can be written as $\vec{a} + \frac{1}{2}\vec{b}$, and is perpendicular to the base of the triangle \vec{b} : $(\vec{a} + \frac{1}{2}\vec{b}) \cdot \vec{b} = \vec{a} \cdot \vec{b} + \frac{1}{2}\vec{b} \cdot \vec{b} = 0$. On the other hand if we square both sides of the equation $|\vec{a} + \vec{b}| = |\vec{a}|$, we get $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = \vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} = \vec{a} \cdot \vec{a} \Rightarrow 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} = 0$ which corresponds to relation in previous equation.



16. It's $\vec{r} - \vec{q} = -(\vec{p} - \vec{q})$, therefore $\vec{r} = 2\vec{q} - \vec{p}$.

17. Unit-circle implies that those are vectors in 2D. $|\vec{r}| = 1$ or $\vec{r} \cdot \vec{r} = 1$ or, with $\vec{r} = [x, y]^T$: $\sqrt{x^2 + y^2} = 1$ or $x^2 + y^2 = 1$, or $\vec{r} = [\cos \varphi, \sin \varphi]^T$.

18. Parallel to the x -axis: $\vec{F}_x = [f_1, 0, 0]^T$, parallel to the y -axis: $\vec{F}_y = [0, f_2, 0]^T$, parallel to the z -axis: $\vec{F}_z = [0, 0, f_3]^T$ 19. No, not necessarily - forces \vec{F}_2 and \vec{F}_3 can cancel each other out.

20. a) and b): \hat{e}_v , c) $-\hat{e}_v$.

21. \vec{u} and \vec{w} are orthogonal to each other.

22. Elementary, my dear Watson!

If there is a v_z , it means that \vec{v} is a 3D-vector with 3 vector components. It couldn't be an abstract n -dimensional vector, because the arrow over it assigns *geometrical* meaning, which reduces the number of possible components to three.

Because \vec{v} is not a null-vector it's length and therefore v_z must have positive value. Furthermore: $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = v_z \Rightarrow v_x^2 + v_y^2 + v_z^2 = v_z^2 \Rightarrow v_x^2 + v_y^2 = 0 \Rightarrow v_x = v_y = 0$ and therefore $\vec{v} = [0, 0, v_z]^T$ - vertical upwards-pointing vector.

But, you know it yourself, Watson: when the height difference between both ends of a rope equals its length, it runs vertically.

23. Their x -coordinates are equal, therefore one of the points is vertically on top of the other. Their distance to each other is therefore module of the difference between their y -coordinates: 3333.

One does not need to calculate any powers or roots!

24. Yes, there is one: the null-vector.

25. No, generally one cannot recognize it if the plane is given in vector equation form. Exception would be if the start vector is 'missing' like in the following case:

$$\vec{\mathcal{P}}(\lambda, \mu) = \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \mu \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \text{ which is equal to } \vec{\mathcal{P}}(\lambda, \mu) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \mu \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

Here one can recognize that the null vector is a start vector and therefore origin is already in the plane. If one sets starting vector to $\vec{a}(2, 3) = [11, 7, 21]^T$ the equation transforms into:

$$\vec{\mathcal{P}}(\lambda, \mu) = \begin{bmatrix} 11 \\ 7 \\ 21 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \mu \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

It's the same plane, however to conclude that it goes through the origin one first needs to solve the over-determined system of linear equations¹⁵.

In point-normal form it is very simple, since after calculating the standard equation the right side d of the equation $ax + by + cz = d$ equals zero (then and only then when plane contains the origin).

¹⁵There are two variables μ and λ , and 3 equations - one for each vector component, thus overdetermined

26. a) \vec{f}_a should point in the same direction as \vec{f} , it means that it can be created by multiplying \vec{f} with a positive factor (scalar). An increase of 20% results in 120% of the start value, therefore the factor is 1.2. With it $\vec{f}_a = 1.2\vec{f} = [7.2, -7.2, 16.8]^T N$.

b) Scalar increase of value of the vector (length) can not somehow be simply added to \vec{f} (like to its x-value, or - because it's nice and symmetrical - in the middle one, or 'evenly' distributed between all 3 components with 6.67 N)!

We first express both vectors through their length and unit vectors:

$$\vec{f} = |\vec{f}| \cdot \hat{u}_f \text{ and } \vec{f}_a = |\vec{f}_a| \cdot \hat{u}_{f_b} ,$$

According to the question both unit vectors are the same. Furthermore $|\vec{f}_b| = |\vec{f}| + 20N$. Because of that:

$$\vec{f}_b = (|\vec{f}| + 20N) \cdot \hat{u}_f = \frac{|\vec{f}| + 20N}{|\vec{f}|} \vec{f} = \frac{\sqrt{36 + 36 + 196} + 20}{\sqrt{36 + 36 + 196}} \begin{bmatrix} 6 \\ -6 \\ 14 \end{bmatrix} N = \frac{\sqrt{67} + 10}{\sqrt{67}} \begin{bmatrix} 6 \\ -6 \\ 14 \end{bmatrix} N = \begin{bmatrix} 13.330 \\ -13.330 \\ 31.104 \end{bmatrix} N.$$

27. $|\lambda \vec{x}| = |\lambda| \cdot |\vec{x}|$, is valid for any real value of λ , therefore:

$$|\vec{a}| = 17 \left| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right| = 17 \cdot \sqrt{1^2 + 2^2} = 17\sqrt{5}$$

$$|\vec{b}| = 209 \left| \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right| = 209 \cdot \sqrt{0^2 + 1^2 + 0^2} = 209$$

$$|\vec{c}| = 19 \left| \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right| = 19 \cdot \sqrt{1^2 + 1^2 + 1^2} = 19\sqrt{3}$$

Practice material

Geometry and vector math¹⁷

Following geometrical problems can be partially solved directly and partially by using vector math. Furthermore this material includes problems from field of statics.

Problems are relatively simple and can be solved through reasonable connections of basic equations.

I suggest to first think about solution on your own. If this does not help one should think even harder and in a case of emergency take it to another level and contemplate about it every spare minute several days in a row.

If one thinks to have found a solution, one can then compare it with the provided one. One might then find out that ones own is even better, or more complicated but correct(why did I choose the complicated path?), or false(why did I make a mistake?).¹⁸This is how people learn.

Problems:

1. Find an equation of plane \mathcal{P} , which is perpendicular to $3x - 2y - z = 106$ and contains origin and point $(1, 1, 6)^T$.
2. Plane \mathcal{P} should go through points $P = (2, 2, 5)^T$, $Q = (4, 7, 3)^T$ and a point R , which is on line $l : \vec{l}(\tau) = [-1, -1, 5]^T + \tau[3, 6, -2]^T$. It should be determined through points P, Q and R . Which point of l is not suitable as R ?
3. Plane \mathcal{P} contains x -axis and point $B = (0, 2, 3)^T$.
 - a) Line l goes through origin and point $Q = (5, 1, 1)^T$. At what angle will it pierce through the plane?
 - b) How far is point $R = (0, 0, 10)^T$ from the plane?
4. a) Two paths lead from point $P = (3, 1, 1)^T$ to points $Q = (7, 2, 4)^T$ and $R = (0, -2, 2)^T$. How wide is the angle they close in?
- b) Construct an equation of line l , which runs parallel to the path between points P and Q , and goes through point R !
5. Robot arm is rotating around point $P(3, 3, 1)^T$ and serves between points $A(5, 1, 2)^T$ and $B(3, 0, 1)^T$. How long is it and which angle is covered when it moves between A and B ?
6. Find a non-trivial¹⁹ vector that is perpendicular to $[1, 2, 3]^T$ and $[2, 1, -4]^T$!
7. A plane is parallel to vectors $[1, 1, 4]^T$ and $[2, 0, -3]^T$. It is pierced through by a line that is running parallel to vector $[-3, 5, 1]^T$. Calculate the angle between this line and normal of the plane!
8. Dangerous object moves in a straight line from point $(35, 18)^T$ to $(2, 120)^T$. Mandatory safety distance is set at 12 units of length. Is it possible to stand safely in point $(22, 60)^T$ when the object moves by?
9. Place B is 23km to the north and $ll\text{km}$ to the east from place A ; lets assume that earth surface is flat and both places are connected through a straight running power line. Power station \mathcal{P} will get built 9km to the north and 1km to the west from A ; it should get connected to the power line with as short connection line as possible. Where should the connection point be and how long is the connecting line?
10. Assign parameter λ value with which vectors $[3, 2, -4]^T$ and $[0, 1, 5]^T + \lambda[2, 5, -1]^T$ are orthogonal to each other!
11. There are five vectors starting from the origin in a plane and each of their dot products is positive. Provide a proof that there is a line going through origin which has all of these vectors on one side! How would one calculate an equation of such line?
12. Points A and B sit on 2 lines that intersect at an angle of 60° . Distance between these points is 31 feet. If we move A 20 feet towards intersection point, then distance between A and B reduces to 21 feet. How far are A and B from intersection point of their lines?
13. A lantern is shining in the middle of a huge square from a 5 m high pole. A pretty tall pedestrian(his hat is at 2 m level from ground)is crossing this square in a straight line 8 m from the foot of lantern pole. In which mathematical curve does the shadow of his hat travel?

¹⁶Date of the first issue of english translation : 29.12.2019

¹⁷Commentary related to the math course and exams this material was partially written for was deliberately excluded by editor

¹⁸or the solution provided in this document is false, that also might happen despite all checks.

¹⁹it means not a null-vector $[0, 0, 0]^T$.

14. Two askew lines $l_1 : \vec{l}_1(\lambda) = \vec{a}_1 + \lambda\vec{p}_1$ and $l_2 : \vec{l}_2(\mu) = \vec{a}_2 + \mu\vec{p}_2$ are given; observer from point \vec{b} , which is not on one of the lines has the impression that they intersect, though in reality they pass each other at some distance. Submit an equation for a line between observer and what he thinks is point of intersection!

15. There are 3 points in a plane that are not on the same line. How can we find out center coordinates and radius of a circle which encompasses all 3 points if we know their coordinates?

16. Plane \mathcal{P} contains points $(1, 2, 1)^T$, $(4, -2, 7)^T$ and $(6, 0, 5)^T$. Submit the intersection line between it and $x - z$ -plane!

17. Find equations for all planes that are parallel to $\mathcal{P} : 6x - 7y - z = 104$ and in distance $\delta = 8$ from it. Submit their equations in form of $ax + by + cz = 1$ ²⁰!

18. Line l passes through points $A = (3, 3, -7)^T$ and $B = (1, -5, 11)^T$.

Find all points on l :

a) whose distance from A is four times higher than that from B or

b) from whom A is four times farther than B .

19. A rope starting from point $X = (x, 0)^T$ on x -axis pulls and is attached at point $P = (19, 34)^T$. Point P is supported by two ropes to $Q = (-3, 105)^T$ and $R = (38, 202)^T$ both pulling in the according directions so that force from P to X is compensated.

What is the allowed interval $a \leq x \leq b$ for x -coordinate of X so that this is possible?

20. Two ropes span from points $P = (2, 5)^T$ and $Q = (-1, 6)^T$ to $R = (5, 11)^T$. Forces pulling towards R equal $f_p = 60N$ and $f_q = 110N$. Total force resulting at point R shall stay same in magnitude and turn it's direction by 4° counter-clockwise. What values should f_p and f_q have in this case?

21. Points $A = (2, 7)^T$, $B = (5, 3)^T$ and $C = (1, -1)^T$ are in the same plane.

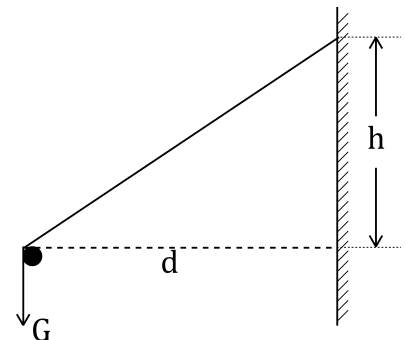
Find all points A' in this plane, where triangles $A'BC$ have the same area as triangle ABC and distance between A' and B is 22.7 !

22. Suspension point $P = [5, 6]^T$ is given. Force of 280N magnitude pulls in direction of point $[13, 2]^T$.

Maximum load at P cannot exceed force of 400N. Another force of 330N magnitude shall be added. At what angle values (measured against positive x -axis) is this allowed?

23. Weight G hangs over a cylinder which is in distance d from a vertical wall. It is supported by a rope that is pinned to a wall at height h over the level of cylinder.

Rope can't resist more than force $F_g < G^{21}$ in horizontal direction. What is the minimum height of h ?

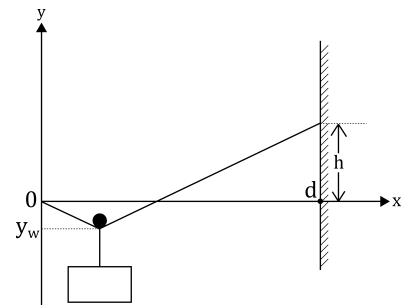


24. Line l runs through origin and splits points $P_1 = (8, 3)^T$ and $P_2 = (15, 20)^T$ so that they stay on opposite sides of the line. Furthermore their distance from l is the same. Provide an equation for l !

25. One end of a rope of length L is pinned at the origin. A wheel with attached weight can move freely across the rope and divides it in two straight parts.

Y -coordinate of the wheel is negative: $y_w < 0$. At which height $y = h$ should the other end of the rope get pinned at a vertical wall running through $x = d < L$ so that this is true?

Hint: At equilibrium both segments of rope have the same slope angle.



²⁰See Supplement \rightarrow Theory \rightarrow Intercept form of a plane

²¹ G is meant as gravitational force pulling the weight towards earth and not its mass

26. 35mm camera captures pictures on $24mm \times 36mm$ film.

Its normal lens covers 45° angle across pictures diagonal. Which angles are covered on horizontal and vertical dimensions when measured through center of the picture?

27. 4 rods of 65cm length each are connected in a circle with hinges so that one can span up a square or rhombus out of it.

This construct measures 130cm in length and is put inside a pipe with 22cm inner radius. A spring connects front and back hinges so that rhombus is formed and other 2 hinges press against walls of the pipe on vertical axis. Spring has a spring constant of 5.4N/mm and stress free length of 82cm.

a) With what amount of force does top and bottom hinges press against walls of the pipe?

b) Position of rhombus inside the pipe is changed so that 2 of the rods run parallel to the walls of the pipe. Which amount of force presses against pipe walls in this case?

28. A straight street should get built on a hillside which is represented as a plane with 29.3° rise against the horizontal. For 100m of distance street should rise 11m in height.

We assign street as a line s , and h as a horizontal line inside of hillside plane; s and h should intersect.

a) Which angle α is enclosed by s and h ?

b) Let s' be a vertical projection of s in the horizontal plane and h' that of h . Which angle β is enclosed by s' and h' ?

29. There are two different points $A = (x_a, y_a)^T$ and $B = (x_b, y_b)^T$ in the plane.

We know of point P in the plane that moves on an unknown(yet) line l in the plane with yet unknown constant velocity v .

At moment $t = 0$ position of P gets measured as $C = (x_c, y_c)^T$. It turns out that *right in this moment* distance d_a from A to P changes with velocity v_a and from B to P with v_b . Find equation of motion of point P with coordinates $x_p(t)$ and $y_p(t)$. A, B and C are not on the same line.

30. Points J, K and L (that are not on the same line) are the end points of 3D-vectors \vec{j}, \vec{k} and \vec{l} starting from origin. Determine set S of all points in space that have the same distance from J, K and L !

31. $\vec{a} = [2, 0, 5]^T$ and $\vec{b} = [-4, -1, 5]^T$ are given. Calculate factors λ and μ so that linear combination²² $\lambda\vec{a} + \mu\vec{b}$ added to $C = [-48, 4, 39]^T$ results in a vector parallel to $\vec{d} = [8, -3, -8]^T$!

32. A rope pulls from point $P = (5, 2)^T$ to $Q = (17, 9)^T$ with a force of 400N. To compensate force at point P two ropes span from P to points $R = (0, -10)^T$ and $S = (-2, -1)^T$ respectively. Which forces must be applied so that total force in P equals zero?

Unit vectors of the three ropes and therefore the force directions equal:

$$\hat{u}_q = \frac{1}{\sqrt{12^2 + 7^2}} \begin{bmatrix} 12 \\ 7 \end{bmatrix} = \begin{bmatrix} 0.863779 \\ 0.503871 \end{bmatrix}, \hat{u}_r = \frac{1}{\sqrt{5^2 + 12^2}} \begin{bmatrix} -5 \\ -12 \end{bmatrix} = \begin{bmatrix} -0.384615 \\ -0.923077 \end{bmatrix},$$

$$\hat{u}_s = \frac{1}{\sqrt{7^2 + 3^2}} \begin{bmatrix} -7 \\ -3 \end{bmatrix} = \begin{bmatrix} -0.919145 \\ -0.393919 \end{bmatrix}$$

Forces f_r and f_s can be calculated by solving system of 2 linear equations resulting from addition of all the forces pulling at point P :

$$400 \begin{bmatrix} 0.863779 \\ 0.503871 \end{bmatrix} + f_r \begin{bmatrix} -0.384615 \\ -0.923077 \end{bmatrix} + f_s \begin{bmatrix} -0.919145 \\ -0.393919 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solution of this system is $f_r = 70.5N$ and $f_s = 346.4N$. Is there an easier way to find the solution?

33. \hat{u}_a and \hat{u}_b are unit vectors of $\vec{a} = [7, 2, 4]^T$ and $\vec{b} = [-5, -1, 3]^T$. Submit $\hat{u}_a \times \hat{u}_b$ in decimal numbers!

34. Why writing \vec{v}^2 for $\vec{v} \times \vec{v}$, and \vec{v}^3 for $\vec{v} \times \vec{v} \times \vec{v}$ is not common for 3D-vectors?

35. Calculate

$$\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \times \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}!$$

36. Vectors $\vec{v} = [6, -4, -5]^T$ and $\vec{w} = [2, y, z]^T$ are given; \hat{u}_v and \hat{u}_w are their corresponding unit vectors. It is known that $\hat{u}_v \cdot \hat{u}_w = 1$. Calculate values of y and z !

37. Force \vec{F}_1 is given. At which angle α should another force of magnitude F_2 pull(from the same point as \vec{F}_1) so that $|\vec{F}_1 + \vec{F}_2| = |\vec{F}_1|$?

38. Line $l: \vec{l}(t) = [1, -1, 4]^T + t[2, 3, -1]^T$ and point $P = (7, 6, 15)^T$ are given. Search for points Q and R on

²²see Supplement→Theory→Linear combination of vectors and for deeper insight into concept of linearity see the supplement question A at the end of this document integrated from other text of W.ROSENHEINRICH called "On solving of problems".

l , so that ΔPQR is equilateral.

39. Line $l : \vec{l}(t) = [1, -1, 4]^T + t[2, 3, -1]^T$ and points $P = (10, 6, 15)^T$ and $Q = (-3, -7, 6)^T \in l$ ²³ are given. Find point R on l , so that ΔPQR has an area of 109.

40. Assign value of a so, that lines $l_1 : \vec{l}_1(\lambda) = [1, 2, 3]^T + \lambda[2, 5, -1]^T$ and $l_2 : \vec{l}_2(\mu) = [7, 0, 3]^T + \mu[6, a, 4]^T$ are in the same plane!

41. Plane \mathcal{P} in 3D-space is given. It has its own coordinate-system S in which point A with space-coordinates $(12, 1, 2)^T$ has system coordinates of $(-2, 3)^T$, and point $B = (4, -3, 8)^T \in \mathcal{P}$ those of $(2, 1)^T$, furthermore is $C = (4, -5, 13)^T$ at S -coordinates of $(4, 2)^T$.

- Find the space-coordinates of origin of S !
- Which coordinates has point $(3, -2, 5)^T \in \mathcal{P}$ in S ?
- What are space-coordinates of point $(5, -6)^T$ of S ?

42. Computer program has two 3D-vectors $\vec{x} = [x_1, x_2, x_3]^T$ and $\vec{y} = [y_1, y_2, y_3]^T$ with integer components x_k and y_k at its disposal. How can it decide if they are parallel or not?

43. Vectors \vec{a} and \vec{b} with $\vec{a} \neq \vec{b}$ are given. For which vectors \vec{r} is the equation $|\vec{r} - \vec{a}| = |\vec{r} - \vec{b}|$ valid?

44. Point-like light source shines from point $L = (2, 5, 22)^T$. Submit an equation describing shadow that line running through $P = (-2, 0, 1)^T$ and $Q = (4, -1, 3)^T$ throws on plane $x + y + 6z = -60$!

45. Plane $\mathcal{P} : 2x - 5y - 2z = 4$ and point $Q = (4, 2, -3)^T \in \mathcal{P}$ are given. Submit an equation for horizontal line in this plane running through Q (with z -axis conventionally pointing upwards).

Is it possible to find a vertical line (in the same manner as the horizontal one) running in plane through Q ?

46. z -axis is as usual pointing upwards in 3D-coordinate system. Point $P(x)$ is a point in plane $14x - 129y + 311z = -1072$, exactly on top of point $(x, 0, 0)^T$ of x -axis. Will point $P(x)$ rise or fall if value of x increases?

47. Two forces of 40N magnitude each pull at the same point. Resulting total force has magnitude of 70N. Which angle is between these two forces?

48. Lines l_1 and l'_1 run through point $P_1 = (2, 6, 1)^T$ and intersect with line $l_1^* : \vec{l}_1^*(t) = [-2, 1, 6]^T + t[1, 1, -3]^T$. Lines l_2 and l'_2 run through point $P_2 = (1, 0, 3)^T$ and intersect with line $l_2^* : \vec{l}_2^*(s) = [3, 3, -2]^T + s[2, 1, 5]^T$.

- Determine l_1 and l_2 knowing that they are parallel to each other!
- l'_1 intersects with l_1^* at point $\vec{P}_1^*(t_i)$ and l'_2 intersects l_2^* at point $\vec{P}_2^*(s_i)$. It is also known that l'_1 and l'_2 intersect. Describe $\vec{P}_2^*(s_i)$ through $\vec{P}_1^*(t_i)$!

49. 2D vector \vec{a} is given. Find vectors \vec{b} and \vec{c} of given lengths $b > 0$ and $c > 0$ so that equation $\vec{b} + \vec{c} = \vec{a}$ is valid. Is there one definite answer?

50. Point $P = (2, 1, 4)^T$ is given. Find point Q on $l : \vec{l}(\lambda) = [7, 2, 9]^T + \lambda[-1, -1, 5]^T$, so that \vec{PQ} has 30° rise against horizontal!

51. Find plane \mathcal{P} with points $(1, 4, 1)^T$ and $(2, 2, 2)^T$ that has 45° rise against horizontal.

²³element of (in this case it means that point Q is on line l)

Solutions:

1. Since \mathcal{P} contains origin, its standard equation must be of type $ax + by + cz = 0$ ²⁴.

Since both planes are orthogonal to each other, dot product of their normal vectors equals zero:

$$\vec{n}_{\mathcal{P}} \cdot \vec{n}_{\mathcal{P}_2} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} = \boxed{3a - 2b - c = 0}$$

Given point must satisfy the standard equation of the plane, therefore $\boxed{a + b + 6c = 0}$. With this we have two homogeneous²⁵ linear equations for three variables. We search for a non-trivial²⁶ solution of this system. First we multiply second equation by 2 and add it to first one to exclude b :

$$\begin{aligned} a + b + 6c &= 0 \mid \cdot 2 (+) \\ 3a - 2b - c &= 0 \\ \Rightarrow 5a + 11c &= 0 \end{aligned}$$

we use the resulting(3) equation to find values of a and c . One can randomly choose one of them and calculate the other. To get an integer result we set $a = 11 \Rightarrow 5 \cdot 11 + 11c = 0 \Rightarrow c = -5$, then we use these values to calculate b : $3 \cdot 11 - 2 \cdot b + 5 = 0 \Rightarrow -2 \cdot b = -38 \Rightarrow b = 19$.

One of the possible solutions is therefore: $11x + 19y - 5z = 0$, all others are multiples of this one.

2. Three points determine a plane, exception is only if they are on the same line (which includes two points being at the same spot).

P and Q are not on l . Because of it the only free option is that line running through P and Q intersects with l . this intersection point would not be suitable as R . To find this point we set both line equations as equal:

$$\vec{P} + s(\vec{Q} - \vec{P}) = \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix} + \tau \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} + s \begin{bmatrix} 4 - 2 \\ 7 - 2 \\ 3 - 5 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix} + \tau \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix} \Rightarrow \begin{aligned} 5s - 6\tau &= -32 \\ s - 3\tau &= -3 \\ -2s + 2\tau &= 0 \end{aligned}$$

this system of linear equations is solvable (although over-determined) with solution $s = \tau = 3$. With it, intersection point of both lines is at coordinates $(8, 17, -1)^T$. This point is not suitable as R .

3. a) Direction vector of x -axis is $\vec{a} = [1, 0, 0]^T$; with it the origin is also in this plane. Since $\vec{B} = [0, 2, 3]^T$ starts out from origin, it is also an element of plane \mathcal{P} . It's normal vector $\vec{n}_{\mathcal{P}}$ can be calculated through cross-product: $\vec{n}_{\mathcal{P}} = \vec{a} \times \vec{B}$.

Simpler: $\vec{n}_{\mathcal{P}}$ must be orthogonal to the x -axis, it is therefore in $y - z$ -plane and of type $\vec{n}_{\mathcal{P}} = [0, p, q]^T$. It's dot product with \vec{B} equals zero: $\vec{n}_{\mathcal{P}} \cdot \vec{B} = 0 = 2p + 3q \Rightarrow \vec{n}_{\mathcal{P}} = [0, 3, -2]$ for example. Now we can calculate angle β between $\vec{n}_{\mathcal{P}}$ and $\vec{Q} = [5, 1, 1]^T$ (that also starts out from origin and therefore the same plane) by using their dot product:

$$|\vec{n}_{\mathcal{P}}||\vec{Q}| \cos \beta = \sqrt{13}\sqrt{27} \cos \beta = \vec{n}_{\mathcal{P}} \cdot \vec{Q} = 5 \cdot 0 + 1 \cdot 3 + 1 \cdot (-2) = 1 \Rightarrow \cos \beta = \frac{1}{\sqrt{13 \cdot 27}} = 0.0534 \Rightarrow \beta = 86.94^\circ$$

Angle between the line and plane is then: $90^\circ - \beta = 3.06^\circ$.

b) We name distance between point R and \mathcal{P} as $\delta_{\mathcal{P}R}$ and vector from origin to R as \vec{r} . Since \mathcal{P} contains the origin, following equation is valid²⁷:

$$|\vec{n}_{\mathcal{P}}| \delta_{\mathcal{P}R} = \pm \vec{n}_{\mathcal{P}} \cdot \vec{r} \Rightarrow \sqrt{0^2 + 3^2 + (-2)^2} \delta_{\mathcal{P}R} = \pm (0 \cdot 0 + 3 \cdot 0 + (-2) \cdot 10) \Rightarrow \sqrt{13} \delta_{\mathcal{P}R} = 20 \Rightarrow \delta_{\mathcal{P}R} = 5.55$$

4. a) The respective vectors are : $\vec{PQ} = \begin{bmatrix} 7 - 3 \\ 2 - 1 \\ 4 - 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$ and $\vec{PR} = \begin{bmatrix} 0 - 3 \\ -2 - 1 \\ 2 - 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}$; We can calculate angle α between both vectors using their dot product:

$$\begin{aligned} \vec{PQ} \cdot \vec{PR} &= 4 \cdot (-3) + 1 \cdot (-3) + 3 \cdot 1 = -12 - 3 + 3 = -12 = |\vec{PQ}||\vec{PR}| \cos \alpha = \\ \sqrt{16 + 1 + 9} \cdot \sqrt{9 + 9 + 1} \cdot \cos \alpha &= \sqrt{26 \cdot 19} \cos \alpha \Rightarrow \cos \alpha = \frac{-12}{\sqrt{494}} = -0.539906 = \cos 122.67^\circ \end{aligned}$$

²⁴If You don't know why, see the solution of question l in "Practice material: vector math" and that of question 25 in "Control questions on vector geometry"

²⁵right sides of equations are equal to zero

²⁶ $a, b, c \neq 0$

²⁷For elaborate explanation see the solution of question m in "Practice material: vector math"

b) $\vec{l}(\tau) = \vec{R} + \tau\vec{PQ} = [0, -2, 2]^T + \tau[4, 1, 3]^T$

5. Points A and B should be in the same distance from P :

$$\delta(P, A) = |\vec{PA}| = \sqrt{(5-3)^2 + (1-3)^2 + (2-1)^2} = \sqrt{9} = 3,$$

$$\delta(P, B) = |\vec{PB}| = \sqrt{(3-3)^2 + (0-3)^2 + (1-1)^2} = \sqrt{9} = 3$$

3 is therefore length of the robot arm. We use properties of dot product to calculate the covered angle φ :

$$\cos \varphi = \frac{\vec{PA} \cdot \vec{PB}}{|\vec{PA}| \cdot |\vec{PB}|} = \frac{\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}}{3 \cdot 3} = \frac{6}{9} \Rightarrow \varphi = 48.190^\circ$$

6. a) We name variables of unknown vector as $[x, y, z]^T$. Since it is orthogonal to both vectors it's dot products with each of them equal zero. We get system of two equations for 3 variables: $x + 2y + 3z = 0, 2x + y - 4z = 0 \Rightarrow -3y - 10z = 0$ One can set $y = 10$ and $z = -3$; x is then -11

b) Calculate cross product of the two vectors:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} = \begin{vmatrix} \hat{u}_x & \hat{u}_y & \hat{u}_z \\ 1 & 2 & 3 \\ 2 & 1 & -4 \end{vmatrix}^{28} = (2 \cdot (-4) - 3 \cdot 1)\hat{u}_x - (1 \cdot (-4) - 3 \cdot 2)\hat{u}_y + (1 \cdot 1 - 2 \cdot 2)\hat{u}_z = \begin{bmatrix} -11 \\ 10 \\ -3 \end{bmatrix}$$

7. Let \vec{n} be the normal vector of the plane:

$$\vec{n} = \begin{vmatrix} \hat{u}_x & \hat{u}_y & \hat{u}_z \\ 1 & 1 & 4 \\ 2 & 0 & -3 \end{vmatrix} = (1 \cdot (-3) - 4 \cdot 0)\hat{u}_x - (4 \cdot 2 - 1 \cdot (-3))\hat{u}_y + (1 \cdot 0 - 1 \cdot 2)\hat{u}_z = \begin{bmatrix} -3 \\ 11 \\ -2 \end{bmatrix}$$

Using properties of dot product we can now calculate the angle α we are searching for:

$$\cos \alpha = \frac{-3 \cdot (-3) + 11 \cdot 5 - 2 \cdot 1}{\sqrt{3^2 + 11^2 + 2^2} \cdot \sqrt{3^2 + 5^2 + 1^2}} = 0.90533 \Rightarrow \alpha = 25.13^\circ$$

Angle at which line pierces the plane is then $90^\circ - 25.13^\circ = 64.87^\circ$. To calculate it directly one could use arcsine function instead of arccosine.

8. Objects line of movement in parametric form can be written as $l(\vec{\tau}) = [35, 18]^T + \tau[-33, 102]^T$; with a normal vector \vec{n} one can transform it into point-normal form $\vec{p} \cdot \vec{n} = d$ with constant d and vector \vec{p} to any point of the line. \vec{n} is orthogonal to direction vector $[-33, 102]^T$, and therefore of type $\lambda[102, 33]^T$ with $\lambda \neq 0$. For calculation of distance it is practical to have $|\vec{n}| = 1$ which means that

$$1 = |\lambda| \cdot |[102, 33]^T| = |\lambda| \sqrt{102^2 + 33^2} = 107.21|\lambda| \Rightarrow \lambda = 1/107.21 = 0.009328.$$

We insert coordinates of point $(35, 18)^T$ into the equation:

$$\vec{p} \cdot \vec{n} = 0.009328 \cdot (35 \cdot 102 + 18 \cdot 33) = 38.84$$

For point $(22, 60)^T$ value of the same calculation is:

$$0.009328 \cdot (22 \cdot 102 + 60 \cdot 33) = 39.40.$$

Difference between these two values $39.40 - 38.84 = 0.56$ is the distance between point and movement line²⁹; it is drastically under the given safety distance of 12.

The same calculation without normalization of normal vector $\vec{n} = [34, 11]^T$ - both of it's components are this time divided by 3 to simplify the calculation.

We use $[22, 60]^T - [35, 18]^T = [-13, 42]^T$:

$$\frac{[-13, 42]^T \cdot [34, 11]^T}{|[34, 11]^T|} = \frac{-13 \cdot 34 + 42 \cdot 11}{\sqrt{34^2 + 11^2}} = \frac{20}{\sqrt{1277}} = 0.56$$

9. We set origin of coordinate system at A ; x - and y -axis point to east and north respectively.

Connection line must run orthogonal to existing line between A and B ; it's line equation must be of form $\vec{l}_c(\tau) = [-1, 9]^T + \tau[p, q]^T$ with $[p, q]^T \perp [11, 23]^T \Rightarrow 11p + 23q = 0$; one possible solution is for example $p = 23, q = -11$. Existing line can be described as $\vec{l}_e(s) = [0, 0]^T + s[11, 23]^T$, intersection point can be

²⁸this determinant is a common calculation scheme for cross product; for elaborate explanation see Supplement→Theory→Calculation of cross product using determinant

²⁹for elaborate proof see supplement→theory→distance between point and a line in plane

calculated by setting both equations equal $[-1, 9]^T + \tau(23, -11)^T = s[11, 23]^T$. From x - and y -components we get following equation system:

$$-1 + 23\tau = 11s \cdot 23(-)$$

$$9 - 11t = 23s \cdot 11$$

$$\Rightarrow -23 + 529\tau - 99 + 121\tau = -122 + 650\tau = 0 \Rightarrow \tau = \frac{122}{650}$$

Intersection point is therefore: $[-1, 9]^T + \frac{122}{650}[23, -11]^T = (3.32, 6.94)^T$

Distance: $\sqrt{3.32 - (-1))^2 + (6.94 - 9)^2} = \sqrt{22.9} = 4.79$

$$10. [3, 2, -4]^T \cdot ([0, 1, 5]^T + \lambda[2, 5, -1]^T) = (3 \cdot 0 + 2 \cdot 1 - 4 \cdot 5) + \lambda(3 \cdot 2 + 2 \cdot 5 + 4 \cdot 1) = -18 + 20\lambda = 0 \Rightarrow \lambda = \frac{18}{20}$$

11. We choose one of these 5 vectors at random and construct a line running orthogonal to it through origin. All five vectors will be on one side of this line since cosine and therefore the dot product is positive between 0° and 90° . If one of vectors would be on other side of this line it's angle with our randomly chosen vector would be between 90° and 180° , thus cosine and dot product negative which is in contrary to problem text. To construct an equation for the line swap axis components of one of the given vectors, change one of the components signs and use it as direction vector of the line. Starting point is the origin.

12. We set coordinate system so that A is on x -axis at coordinate value $z \geq 20$; second line with B in it is in first quadrant and point B has coordinates $(x, y)^T$. Intersection point of both lines is at origin. Using PYTHAGORAS and trigonometry we get:

$$(x - z)^2 + y^2 = 31^2 = 961$$

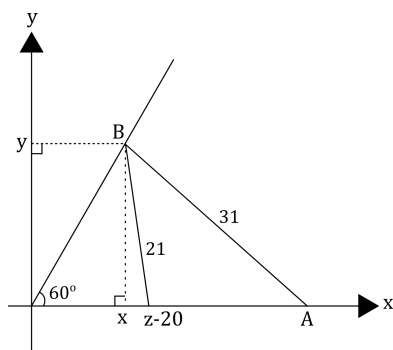
$$(x - (z - 20))^2 + y^2 = 21^2 = 441$$

$$y = \tan 60^\circ \cdot x = \sqrt{3}x$$

By expanding second equation we get: $(x - z)^2 + 40(x - z) + 400 + y^2 = 441$, and by subtracting first equation from it we get: $40(x - z) + 400 = -520 \Rightarrow x - z = -\frac{920}{40} = -23$ We use the value of $x - z$ in first equation:

$$23^2 + y^2 = 961 \Rightarrow y^2 = 432 \Rightarrow y = 12\sqrt{3} \text{ and } x = 12. \text{ Since } x - z = -23 \Rightarrow z = 12 + 23 = 35$$

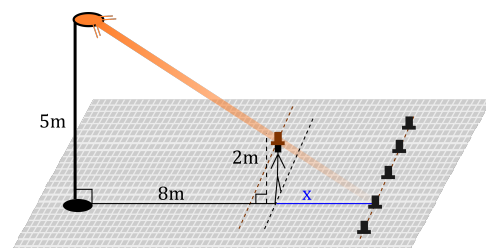
Distance of point A from origin is therefore 35, for point B it is $\sqrt{12^2 + 3 \cdot 12^2} = 24$.



13. By ignoring up and down motion during each individual step one can conclude that head and therefore hat moves in a straight line in space. This line and point of bulb of lantern define a plane in space that intersects with ground plane of the square. This intersection line is the one on which shadow of the hat is moving (although with non-constant velocity). It is parallel to the one that pedestrian is walking on. We can use THALES theorem to calculate its distance from the lantern:

$$\frac{5}{2} = \frac{8+x}{x} \Rightarrow 5x = 16 + 2x \Rightarrow x = 5\frac{1}{3}$$

Therefore distance is $8 + 5\frac{1}{3} = 13.33$ meters.



14. Point \vec{b} and l_1 set up a plane P_1 , point \vec{b} and l_2 plane P_2 . Searched line is intersection line of these two planes. Its direction vector \vec{q} is therefore orthogonal to normal vectors of P_1 and P_2 .

Since $\vec{p}_1 \nparallel \vec{a}_1 - \vec{b}$, one can calculate $\vec{n}_1 = \vec{p}_1 \times (\vec{a}_1 - \vec{b})$ for normal vector of P_1 , and in the same fashion $\vec{n}_2 =$

$\vec{p}_2 \times (\vec{a}_2 - \vec{b})$ for the other plane. For equation of resulting line we get: $\vec{l}_{res}(\nu) = \vec{b} + \nu[\vec{p}_1 \times (\vec{a}_1 - \vec{b})] \times [\vec{p}_2 \times (\vec{a}_2 - \vec{b})]$

15. There is a way of solving this problem by using arithmetic only: we name center coordinates a and b and radius of the circle as r . With coordinates of the three points $x_i, y_i, i = 1, 2, 3$ we get a system of 3 non-linear equations for the 3 unknown variables a, b and r :

$$(x_i - a)^2 + (y_i - b)^2 = r^2, \quad i = 1, 2, 3$$

It can be solved with reasonable effort.

From geometrical perspective following reasoning can be applied:

Path between two of these points must be chord of the searched circle. Perpendicular bisector of this chord runs through center of circle. Therefore we assign a line l_1 which runs through middle point of path between points P_1 and P_2 , and is perpendicular to it. Vector $\vec{n}_1 = [y_1 - y_2, x_2 - x_1]^T$ is therefore a normal vector of l_1 and point $P_{1/2} = \frac{1}{2}(x_1 + x_2, y_1 + y_2)^T$ is on line l_1 . We can now calculate equation of l_1 in point-normal form:

$$\vec{n}_1 \cdot \vec{P}_{1/2} = (y_1 - y_2) \cdot \frac{1}{2}(x_1 + x_2) + (x_2 - x_1) \cdot \frac{1}{2}(y_1 + y_2)$$

In the same fashion we connect first point with the third (or 2nd and 3rd) to get an equation for line l_2 :

$$\vec{n}_2 \cdot \vec{P}_{1/3} = (y_1 - y_3) \cdot \frac{1}{2}(x_1 + x_3) + (x_3 - x_1) \cdot \frac{1}{2}(y_1 + y_3)$$

Point where these two lines cross is center of the circle. We can assign it coordinates $x = a$ and $y = b$ and insert it in two previous equations to create a linear equation system:

$$\begin{aligned} (y_1 - y_2) \cdot a + (x_2 - x_1) \cdot b &= (y_1 - y_2) \cdot \frac{1}{2}(x_1 + x_2) + (x_2 - x_1) \cdot \frac{1}{2}(y_1 + y_2) \\ (y_1 - y_3) \cdot a + (x_3 - x_1) \cdot b &= (y_1 - y_3) \cdot \frac{1}{2}(x_1 + x_3) + (x_3 - x_1) \cdot \frac{1}{2}(y_1 + y_3) \end{aligned}$$

After solving it one can then calculate radius of circle as:

$$r = \sqrt{(x_1 - a)^2 + (y_1 - b)^2}.$$

This kind of question can be also solved as search for circumcircle of a triangle.

16. 1. Method:

$$\mathcal{P} : \vec{p}(s, t) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix} + t \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$$

$x - z$ -plane is set through $y = 0$, which means that middle component stays void. It binds value of s to t : $0 = 2 - 4s - 2t \Rightarrow t = 1 - 2s$

Equation of the line is therefore:

$$\vec{l}(s) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix} + (1 - 2s) \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 5 \end{bmatrix} + s \begin{bmatrix} -7 \\ 0 \\ -2 \end{bmatrix}$$

Reduced to $x - z$ -plane ($2D$):

$$\vec{l}(s) = \begin{bmatrix} 6 \\ 5 \end{bmatrix} + s \begin{bmatrix} -7 \\ -2 \end{bmatrix}$$

With normal vector $\vec{n} = [2, -7]^T$ using dot product we can calculate standard form of the line:

$$\begin{bmatrix} 6 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -7 \end{bmatrix} = -23 \Rightarrow 2x - 7z = -23$$

Written explicitly as a linear function:

$$z = \frac{2}{7}x + \frac{23}{7}$$

2.Method: We construct normal vector of \mathcal{P} from direction vectors $[3, -4, 6]^T$ and $[5, -2, 4]^T$:

$$\begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix} \times \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 18 \\ 14 \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 \\ 9 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 23 \Rightarrow \mathcal{P} : -2x + 18y + 7z = 23$$

One lands in $x - z$ -plane by setting $y = 0 \Rightarrow -2x + 7z = 23$.

3.Method: We set two lines l_1 and l_2 running through two pairs of points:

$$\vec{l}_1(s) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix}, \quad \vec{l}_2(\tau) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \tau \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$$

Through setting y-components equal to zero we can calculate the respective intersection points with $x - z$ -plane : $2 - 4s = 0$ and $2 - 2\tau = 0$, therefore $s_{intersect} = 1/2$ and $\tau_{intersect} = 1$

By setting these two values in their line equations we get intersection points $(2.5, 0, 4)^T$ and $(6, 0, 5)^T$ with $x - z$ -plane. Line running through these points is the one we are searching for:

$$\vec{l}(i) = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} + i \begin{bmatrix} 3.5 \\ 1 \end{bmatrix}$$

17. One can search for such a plane by assigning its equation as $6x - 7y - z = d$, it is parallel to the given plane since their normal vectors are the same.

Since distance from given plane to searched plane is set we get:

$$\delta = 8 = \frac{|d - 104|}{|\vec{n}|} = \frac{|d - 104|}{\sqrt{6^2 + 7^2 + 1^2}} = \frac{|d - 104|}{\sqrt{86}},$$

therefore $|d - 104| = 8\sqrt{86}$. There are two options:

a) if $d - 104 = 8\sqrt{86}$, then $d = 104 + 8\sqrt{86}$, and one gets the plane equation:

$$6x - 7y - z = 104 + 8\sqrt{86} \Rightarrow \frac{6}{104 + 8\sqrt{86}}x - \frac{7}{104 + 8\sqrt{86}}y - \frac{1}{104 + 8\sqrt{86}}z = 1$$

b) if $-(d - 104) = 8\sqrt{86}$, then $d = 104 - 8\sqrt{86}$, and one gets the plane equation:

$$6x - 7y - z = 104 - 8\sqrt{86} \Rightarrow \frac{6}{104 - 8\sqrt{86}}x - \frac{7}{104 - 8\sqrt{86}}y - \frac{1}{104 - 8\sqrt{86}}z = 1$$

18. Line equation:

$$\vec{l}(\tau) = \begin{bmatrix} 3 \\ 3 \\ -7 \end{bmatrix} + \tau \begin{bmatrix} -2 \\ -8 \\ 18 \end{bmatrix}$$

a) We can multiply direction vector with $\tau = 4$ or $\tau = -4$ so its length gets 4 times as big in the particular direction. With it we get points $(-5, -29, 65)^T$ and $(11, 35, -79)^T$.

b) Condition is satisfied either if point lies between A and B , or outside on side of B .

1. case: $\tau = 0.8$ because $0.2 + 0.8 = 1$ and $0.2 : 0.8 = 1 : 4$, therefore $(1.4, -3.4, 7.4)^T$.

2. case: $\tau = 4/3$ because $4/3 - 1/3 = 1$ and $(1/3) : (4/3) = 1 : 4$, therefore $(0.333, -7.667, 17)^T$.

19. a is intersection point of line running through P and R with x -axis. We can use THALES theorem to calculate line function and its intersection point with x -axis:

$$\frac{34 - y_{30}}{19 - x} \Rightarrow y = 34 + \frac{202 - 34}{38 - 19}(x - 19) = 34 + \frac{168}{19}(x - 19) \Rightarrow a = \frac{1273}{84} = 15.155$$

On the other hand b is intersection point with x -axis of the line running through P and Q :

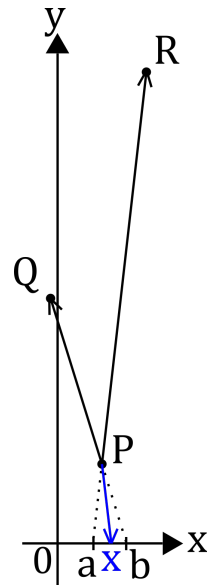
$$\frac{105 - 34}{-3 - 19} = \frac{34 - y}{x - 19} \Rightarrow y = 34 + \frac{105 - 34}{-3 - 19}(x - 19) = 34 - \frac{71}{22}(x - 19) \Rightarrow b = \frac{2097}{71} = 29.535$$

Formal solution by using vector equation:

Two forces to Q and R cancel the force to X . Absolute value of these forces is irrelevant for this solution.

One could use unit vectors but it is not necessary. Let us write out the complete vector equation(their directions are set according to their respective force directions):

$$\begin{bmatrix} x - 19 \\ -34 \end{bmatrix} = -\lambda \begin{bmatrix} -22 \\ 71 \end{bmatrix} - \mu \begin{bmatrix} 19 \\ 168 \end{bmatrix}$$



³⁰One can also calculate a directly using $\frac{202 - 34}{38 - 19} = \frac{34}{19 - a}$, W.Rosenheinrich chose to use x and y to write an explicit function

We derive explicit equations for λ and μ from component equations:

$$\begin{aligned} x - 19 &= 22\lambda - 19\mu & -34 &= -71\lambda - 168\mu \\ \lambda &= -\frac{71}{5045}x + \frac{2097}{5045} & \mu &= -\frac{168}{5045}x + \frac{2546}{5045} \end{aligned}$$

Ropes can only transfer tension forces, therefore their directions must stay the same. Because of it parameters λ and μ must stay positive:

$$\begin{aligned} \mu &\geq 0 & \lambda &\geq 0 \\ -168x + 2546 &\geq 0 & -71x + 2097 &\geq 0 \\ x &\geq 2546/168 & x &\leq 2097/71 \end{aligned}$$

20. We approach the problem with following method: change of both forces can be described through factors λ and μ . New forces then equal λf_p and μf_q . By looking at the problem from geometrical perspective it is easy to see that f_q must get 'tighter' and f_p 'looser'. From it we can derive that $\mu > 1$ and $\lambda < 1$ (but $\lambda > 0$ — since ropes cannot propagate pressure forces). We name unit vector in direction from R to P \hat{u}_p , and that to Q as \hat{u}_q . Then following expressions are true (unit of measurement N excluded):

$$\begin{aligned} |60\hat{u}_p + 110\hat{u}_q| &= |60\lambda\hat{u}_p + 110\mu\hat{u}_q| \\ \text{and } [60\hat{u}_p + 110\hat{u}_q] \cdot [60\lambda\hat{u}_p + 110\mu\hat{u}_q] &= \cos 4^\circ \cdot |60\hat{u}_p + 110\hat{u}_q|^2. \end{aligned}$$

First we determine unit vectors: $\vec{PR} = [3, 6]^T \Rightarrow \hat{u}_p = \frac{1}{\sqrt{5}}[1, 2]^T$. In

similar manner $\vec{QR} = [6, 5]^T \Rightarrow \hat{u}_q = \frac{1}{\sqrt{61}}[6, 5]^T$.

With real numbers:

$$\left| 60 \begin{bmatrix} 0.447214 \\ 0.894427 \end{bmatrix} + 110 \begin{bmatrix} 0.768221 \\ 0.640184 \end{bmatrix} \right| = \left| \begin{bmatrix} 111.3372 \\ 124.0859 \end{bmatrix} \right| = \sqrt{27793.277} = 166.713,$$

from first equation we then get:

$$\begin{aligned} \left| 60\lambda \begin{bmatrix} 0.447 \\ 0.894 \end{bmatrix} + 110\mu \begin{bmatrix} 0.768 \\ 0.640 \end{bmatrix} \right| &= 27793.277 \Rightarrow \left| \begin{bmatrix} 26.833\lambda \\ 53.665\lambda \end{bmatrix} + \begin{bmatrix} 84.504\mu \\ 70.420\mu \end{bmatrix} \right| = 27793.277 \Rightarrow \\ (26.833\lambda + 84.504\mu)^2 + (53.665\lambda + 70.420\mu)^2 &= 27793.277 \end{aligned}$$

or written out:

$$3600\lambda^2 + 12093.277\lambda\mu + 12100\mu^2 - 27793.277 = 0$$

By setting numbers into the dot product equation we get:

$$\begin{bmatrix} 111.337 \\ 124.086 \end{bmatrix} \cdot \begin{bmatrix} 26.833\lambda + 84.504\mu \\ 53.665\lambda + 70.420\mu \end{bmatrix} = 9646.638\lambda + 18146.638\mu = \cos 4^\circ \cdot 166.713^2 = 27725.574$$

From $9646.638\lambda + 18146.638\mu = 27725.574$ we get $\mu = -0.532\lambda + 1.528$ which we can then use for a substitute in first equation:

$$0 = 3600\lambda^2 + 12093.277\lambda(-0.532\lambda + 1.528) + 12100(-0.532\lambda + 1.528)^2 - 27793.277 = 590.652\lambda^2 + 452.537\lambda - 11784259$$

Quadratic equation has two solutions: $\lambda_1 = 1.476$ and $\lambda_2 = 0.519$.

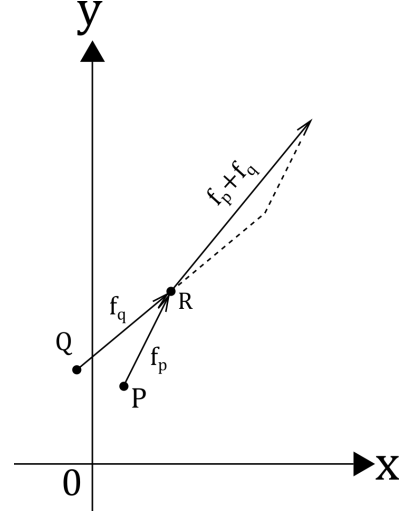
As we found out previously $0 < \lambda < 1$, therefore $\lambda = \lambda_2$ and $\mu = 1.252 > 1$.

New forces are then:

$$|0.519 \cdot 60 \begin{bmatrix} 0.447 \\ 0.894 \end{bmatrix}| = \left| \begin{bmatrix} 13.928 \\ 27.855 \end{bmatrix} \right| = 31.143$$

and

$$|1.252 \cdot 110 \begin{bmatrix} 0.768 \\ 0.640 \end{bmatrix}| = \left| \begin{bmatrix} 105.794 \\ 88.162 \end{bmatrix} \right| = 137.713.$$



Test:

$$\left| \begin{bmatrix} 13.928 \\ 27.855 \end{bmatrix} + \begin{bmatrix} 105.794 \\ 88.162 \end{bmatrix} \right| = \left| \begin{bmatrix} 119.722 \\ 116.017 \end{bmatrix} \right| = 166.713$$

- magnitude of the resulting force has not changed.

Let us test the dot product too:

$$\begin{bmatrix} 111.337 \\ 124.086 \end{bmatrix} \cdot \begin{bmatrix} 105.794 \\ 88.162 \end{bmatrix} = 27725.574 \text{ and } 27793.277 \cdot \cos 4^\circ = 27725.574 - \text{also true.}$$

One could also set 2 new forces as unknown variables and operate with 4 unknown coefficients. Method covered in this solution using linear combination of old vectors is (in my opinion) more effective.

21. Triangles $A'BC$ share their baselines BC with triangle ABC . Since their areas are the same their height must be the same too. Distance of point A' to the line l running through points B and C must therefore be the same as distance between A to l .

Because $\vec{BC} = -[4, 4]^T$ normal vector of line l can be (for example) set as $\vec{n} = [1, -1]^T$ and therefore $\vec{n} \cdot \vec{B} = 2$.

We set coordinates of A' as variables $A' = [x, y]^T$, for distance between l and A' we then get: $x - y = 2$:

$$\left| \frac{1 \cdot x - 1 \cdot y - 2}{\sqrt{2}} \right| = \left| \frac{1 \cdot 2 - 1 \cdot 7 - 2}{\sqrt{2}} \right| \Leftrightarrow |x - y - 2| = 7.$$

There are two cases: It is either $y = x - 9$ or $y = x + 5$. (In middle of these two parallel lines runs $y = x - 2$. It contains B and C , and is therefore l)

Now we can use distance to B : $\sqrt{(x - 5)^2 + (y - 3)^2} = 22.7$, therefore $(x - 5)^2 + (y - 3)^2 = 22.7^2$.

If $y = x - 9$, then we get $(x - 5)^2 + (x - 12)^2 = 22.7^2$ or $2x^2 - 34x - 346.29 = 0$. By solving the equation we get first pair of points $A'_1 = (24.165, 15.165)^T$ and $A'_2 = (-7.165, -16.165)^T$.

In case of $y = x + 5$, we get $(x - 5)^2 + (x + 2)^2 = 22.7^2 \Rightarrow 2x^2 - 6x - 486.29 = 0$ with solutions $A'_3 = (-14.165, -9.165)^T$ and $A'_4 = (17.165, 22.165)^T$.

22. Already existing force vector has direction of $[13, 2]^T - [5, 6]^T = [8, -4]^T$, with $||[8, -4]^T|| = \sqrt{80}$ its unit vector equals $[0.894, -0.447]^T$. After multiplying it with magnitude of 280N we get following representation:

$$\vec{F}_1 = 280N \begin{bmatrix} 0.894 \\ -0.447 \end{bmatrix} = \begin{bmatrix} 250.44 \\ -125.22 \end{bmatrix} N$$

We use angle φ to describe unit vector of the second force so that it is the angle between positive x -axis and force vector - $\hat{u}_{F_2} = [\cos \varphi, \sin \varphi]^T$. Resulting total force at point P is then:

$$\vec{F}_{tot} = \begin{bmatrix} 250.44 \\ -125.22 \end{bmatrix} N + 330N \begin{bmatrix} \cos \varphi \\ \sin \varphi \end{bmatrix}$$

its length(magnitude):

$$\begin{aligned} |\vec{F}_{tot}| &= \sqrt{(250.44 + 330 \cos \varphi)^2 + (-125.22 + 330 \sin \varphi)^2} N = \\ &= \sqrt{280^2 + 330^2 + 2 \cdot 250.44 \cdot 330 \cdot \cos \varphi - 2 \cdot 125.22 \cdot 330 \cdot \sin \varphi} N = \\ &= \sqrt{187300 + 660 \cdot (250.44 \cdot \cos \varphi - 125.22 \cdot \sin \varphi)} N. \end{aligned}$$

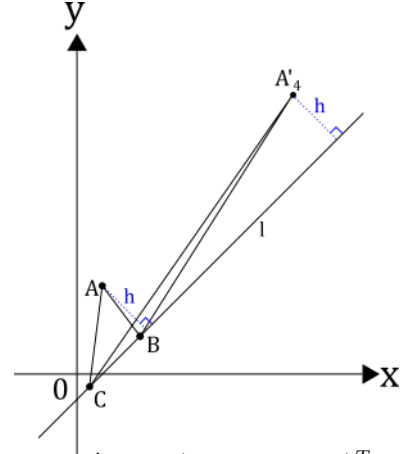
To simplify the root function we express vector $[250.44, 125.22]^T$ in polar coordinates: $[r \cos \alpha, r \sin \alpha]^T$ with $r = 280$ and $\tan \alpha = r \sin \alpha / r \cos \alpha = 0.5$, therefore $\alpha = 26.565^\circ$.

We can now insert these values in previous equation and get:

$$\begin{aligned} |\vec{F}_{tot}| &= \sqrt{187300 + 660 \cdot 280(\cos 26.565^\circ \cos \varphi - \sin 26.565^\circ \cdot \sin \varphi)} N = \\ &= \sqrt{187300 + 184800(\cos 26.565^\circ \cos \varphi - \sin 26.565^\circ \cdot \sin \varphi)} N = \sqrt{187300 + 184800 \cos(\varphi + 26.565^\circ)} N \end{aligned} \quad ^{31}$$

Since expression under the square root is always positive (as values of cosine function never exceed ± 1) we can square it.

³¹last transformation is done by using cosine angle addition identity: $\cos(\alpha + \beta) = \cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta$; for elaborate proof of this identity see Supplement \rightarrow Theory \rightarrow Angle addition identities



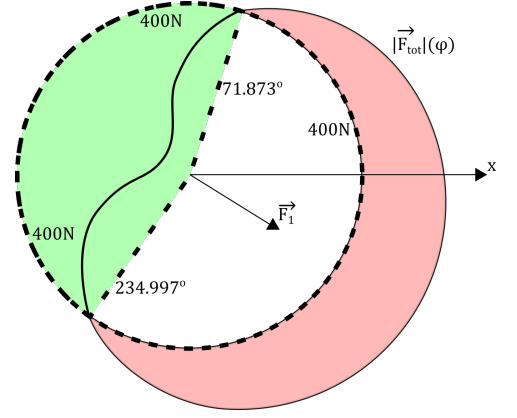
By setting total force equal to its allowed maximum we get:

$$\begin{aligned} 400N &= \sqrt{187300 + 184800 \cos(\varphi + 26.565^\circ)}N \\ 160000 &= 187300 + 184800 \cdot \cos(\varphi + 26.565^\circ) \\ \cos(\varphi + 26.565^\circ) &= -0.147727 = \cos 98.438^\circ \end{aligned}$$

Since cosine has two angles that lead to the same function value in 1st and 4th quadrant respectively we get two angles: $\varphi = 98.438^\circ - 26.565^\circ = 71.873^\circ$, or $\varphi = -98.438^\circ - 26.565^\circ = -125.003^\circ = 234.997^\circ$.

Total force exceeds allowed value somewhere between these two angle values, more specifically the part that contains direction of \vec{F}_1 . Because of that: $71.873^\circ \leq \varphi \leq 234.997^\circ$ ³²

Graph on the right shows curve of total force with point P as center point of the 400N circle area; minimal value of total force is in case where \vec{F}_2 is in exact opposite direction of \vec{F}_1 and therefore $|\vec{F}_{tot}| = 330N - 280N = 50N$.



³²These values are somewhat incorrect because of rounding errors, higher precision calculation gets us 71.930° and 234.940° .

23. We split the force \vec{G} pulling at the point where rope is pinned to the wall in horizontal \vec{F}_h and vertical \vec{F}_v components. Vertical component is easily absorbed by the wall. Vector \vec{G} is parallel to direction of rope from pinpoint to cylinder therefore with scalar λ :

$$\vec{F}_v = \lambda \vec{h} \quad \vec{F}_h = \lambda \vec{d}$$

From this we get:

$$h = \frac{F_v}{\lambda} = \frac{F_v}{F_h} d = \frac{\sqrt{G^2 - F_g^2}}{F_g} d$$

whereby we already inserted the limit value F_g in place of F_h . F_v value is calculated using PYTHAGORAS.

Check: If $F_g = G$, then we get $h = 0$ which is obviously true, and if $F_g \rightarrow 0$ then $h \rightarrow \infty$ which does correspond to our expectations that in this case rope should run almost vertical.

24. We choose unit vector $\hat{u}_n = [\cos \varphi, \sin \varphi]^T$ as normal vector of the line. it is obvious that to find line equation it is enough to calculate its rise angle $\varphi + \frac{\pi}{2}$.

Standard equation of the line is then $x \cos \varphi + y \sin \varphi = 0$. Right hand side is zero since the line travels through origin.

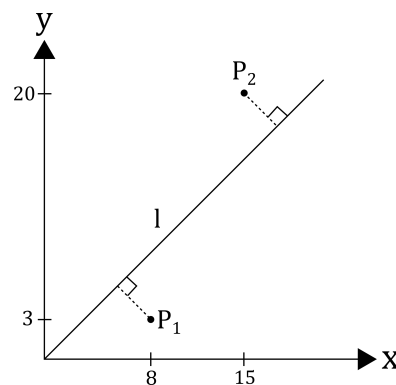
For distance of both points from line we then get:

$$8 \cos \varphi + 3 \sin \varphi = -(15 \cos \varphi + 20 \sin \varphi)$$

We used the fact that \hat{u}_n is a unit vector and minus sign on right hand side shows that P_1 and P_2 is on different sides of the line³³. Now we can calculate angle φ :

$$23 \cos \varphi + 23 \sin \varphi = 0 \rightarrow \varphi = -\pi/4$$

Line equation $x \cos(-\pi/4) + y \sin(-\pi/4) = 0 = x \frac{\sqrt{2}}{2} - y \frac{\sqrt{2}}{2}$ multiplied with $\sqrt{2}$ results in $x - y = 0$ or $y = x$.



25. We assign to the wheel yet unknown x-coordinate x_w , then, because of similarity of corresponding triangles we get: $\frac{-y_w}{\sqrt{x_w^2 + y_w^2}} = \frac{h - y_w}{\sqrt{(d - x_w)^2 + (h - y_w)^2}}$

and because of length condition we have: $\sqrt{x_w^2 + y_w^2} + \sqrt{(d - x_w)^2 + (h - y_w)^2} = L$.

Using these two equations we can calculate value of variable h . Furthermore we can calculate variable x_w . But this kind of calculation is troublesome.

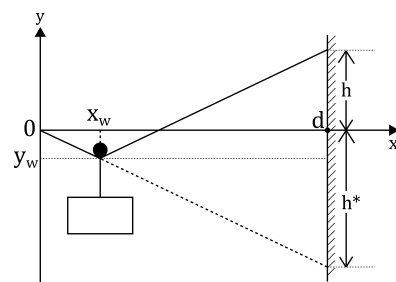
A simpler solution can be won through mirroring the rising part of rope downwards in respect to $y = y_w$ -axis. Since both rise angles are the same we get a line through left part of the rope meeting wall at height $y = h^*$.

It is obvious that y_w is mean average of h and h^* , thus $y_w = \frac{h + h^*}{2} \Rightarrow h = 2y_w - h^*$. (congruent triangles, two sides and angle between them are the same).

Using PYTHAGORAS we get: $d^2 + (h^*)^2 = L^2$ or $h^* = \sqrt{L^2 - d^2}$.

Result: $h = \sqrt{L^2 - d^2} + 2y_w$

To test it think of perfectly firm rope: $L = d$; in this case we can only have $y_w = 0$ with the corresponding height $h = 0$.



³³for more information see Supplement→Theory→Distance between point and a line in plane

26. Let P be an imaginary point in front or back of the picture from which one sees the corners of picture in 45° angle. This point should be on normal that goes through the center of picture; we name distance between it and picture plane d_p .

When looking from this point middle of picture and corner covers angle of 22.5° . Let $2h$ be height of picture, $2w$ -width and $2d$ its diagonals length. Proportions between those three are then $h : w : d = 24 : 36 : \sqrt{24^2 + 36^2} = 2 : 3 : \sqrt{13}$.

Let α be the width angle and β height angle (from P), then from relationships of right triangle we get:

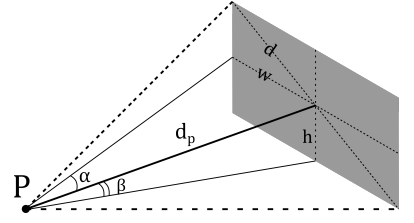
$$\sin 22.5^\circ = \frac{d}{d_p}, \quad \sin \frac{\alpha}{2} = \frac{w}{d_p}, \quad \sin \frac{\beta}{2} = \frac{h}{d_p}$$

From this we get:

$$\alpha = 2 \arcsin \frac{w}{d_p} = 2 \arcsin \frac{w \sin 22.5^\circ}{d} = 2 \arcsin \frac{3 \sin 22.5^\circ}{\sqrt{13}} = 37.13^\circ$$

$$\beta = 2 \arcsin \frac{h}{d_p} = 2 \arcsin \frac{h \sin 22.5^\circ}{d} = 2 \arcsin \frac{2 \sin 22.5^\circ}{\sqrt{13}} = 24.51^\circ$$

27. a) Rhombus can be split in 4 congruent right triangles. Their hypotenuse is 65cm long, opposite is $22\text{cm}/2 = 11\text{cm}$, and adjacent $\sqrt{65^2 - 11^2}\text{cm} = 64.0625\text{cm}$ respectively. Spring is therefore stretched to $2 \cdot 64.0625\text{cm} = 128.125\text{cm}$ in length, length difference to springs normal length is therefore $128.125\text{cm} - 82\text{cm} = 46.125\text{cm} = 461.25\text{mm}$. Spring generates a force of $461.25\text{mm} \cdot 5.4\text{N/mm} = 2491\text{N}$ in direction of pipes center axis.



This force must be compensated by pairs of rods. Force applied to one of the rods is composed by half of the total spring force - 1245N (other half is applied to opposite rod) - in horizontal direction and (yet) unknown vertical component. Since position of the rod is set these force components have the same proportions as adjacent and opposite side of the triangle, therefore:

$$\frac{|\vec{F}_{vert}|}{1245\text{N}} = \frac{11\text{cm}}{64.06\text{cm}} \Rightarrow F_{vert} = \frac{11\text{cm} \cdot 1245\text{N}}{64.06\text{cm}} = 213,8\text{N}$$

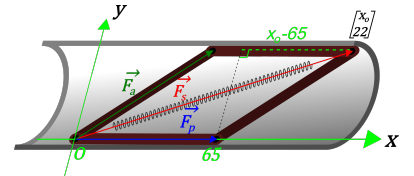
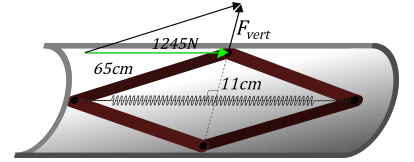
A pair of rods presses with double of this force: $2F_{vert} = 427.6\text{N}$

b) We assign a coordinate system with origin as the lower left hinge of rhombus. One of the rods running parallel to wall of the pipe has the coordinates of x -axis. Its right hinge has coordinates of $(65, 0)^T$, upper right corner has coordinates of $(x_o, 22)^T$. Using PYTHAGORAS we get:

$$(x_o - 65)^2 + 22^2 = 65^2 \quad x_o^2 - 130x_o + 65^2 + 22^2 = 65^2$$

$$x_o^2 - 130x_o + 484 = 0 \quad \mathcal{D} = \frac{p^2}{4} - q = \frac{130^2}{4} - 484 = 3741$$

$$x_{o1} = \frac{130}{2} + \sqrt{3741} = 126.16$$



Length of vector $\vec{F}_s = [126.16, 22]^T$ is equal to the length of spring:

$$|\vec{F}_s| = \sqrt{126.16^2 + 22^2} = 128.06\text{cm}$$

Resulting force is then $F_{spring} = \Delta l \cdot 5.4\text{N/mm} = (128.06\text{mm} - 820\text{mm})5.4\text{N/mm} = 2487\text{N}$ - a little bit less than in a).

Force at origin is split between rods running parallel and askew to the pipe, we can assign these forces magnitudes of F_p and F_a . For equilibrium of forces at origin we then get:

$$F_{spring} \cdot \hat{u}_{F_s} = F_p \cdot \hat{u}_{F_p} + F_a \cdot \hat{u}_{F_a}$$

$$\frac{2487\text{N}}{128.06} \begin{bmatrix} 126.16 \\ 22 \end{bmatrix} = \begin{bmatrix} F_p \\ 0 \end{bmatrix} + \frac{F_a}{65} \begin{bmatrix} 61.16 \\ 22 \end{bmatrix}$$

We are only interested in y-component of this equation:

$$\frac{22}{128.06} \cdot 2487\text{N} = \frac{22}{65} \cdot F_a \Leftrightarrow F_a = \frac{65}{128.06} \cdot 2487\text{N} \Leftrightarrow \frac{22}{65} \cdot \frac{65}{128.06} \cdot 2487\text{N} = 427.3\text{N}$$

Pressure force on walls of the pipe is $427.3N$ - no significant change in comparison to a).

28. The question can be solved by simple reasoning. In this solution however we will choose standard method by using a coordinate system. Let h be x -axis of this system and its intersection point with s be systems origin; y -axis is running horizontally inside of mountain and z -axis is as usual pointing vertically upwards. If we travel $100m$ on the street in the way up, we get at point P with coordinates $(x_p, y_p, 11)^T$ (Unit of measurement being meters. Since we travelled $100m$ then length of \vec{P} is $100m$:

$$x_p^2 + y_p^2 + 11^2 = 100^2$$

Path from $P_1 = (x_p, 0, 0)^T$ to P has maximal rise in plane of the hill. We can use this point and the right triangle it spans up together with P and its projection on the horizontal $x - y$ -plane $P_{xy} = (x_p, y_p, 0)$, since it contains the given rise angle of 29.3° :

$$\frac{z_P - z_{P_{xy}}}{y_{P_{xy}} - y_{P_1}} = \tan 29.3^\circ \Rightarrow \frac{11 - 0}{y_p - 0} = \tan 29.3^\circ \Rightarrow y_p = 19.602$$

We can now insert this value in previous equation: $x_p^2 + 19.602^2 + 11^2 = 100^2 \Rightarrow x_p = 97.441$. $\vec{P} = (97.441, 19.602, 11)^T$ can now be used as direction vector of s , and $\vec{P}' = (97.441, 19.602, 0)^T$ as that of s' .

Practical direction vector of h and h' would be $\hat{u}_z = (1, 0, 0)^T$.

a) By using properties of dot product we get \vec{P} :

$$\cos \alpha = \frac{\vec{P} \cdot \hat{u}_z}{|\vec{P}| \cdot |\hat{u}_z|} = \frac{97.441 \cdot 1}{100 \cdot 1} \Rightarrow \alpha = 13.0^\circ$$

b) For the projection we get:

$$\cos \beta = \frac{\vec{P}' \cdot \hat{u}_z}{|\vec{P}'| \cdot |\hat{u}_z|} = \frac{97.441 \cdot 1}{\sqrt{100^2 - 11^2} \cdot 1} \Rightarrow \beta = 11.4^\circ$$

29. Velocity vector \vec{v} of P has constant, time-independent magnitude, and since the point moves in a straight line its direction is constant as well.

Therefore $\vec{v} = [v_x, v_y]^T$ is a constant vector.

We use C as a starting point, with it:

$$\begin{bmatrix} x_p(t) \\ y_p(t) \end{bmatrix} = \begin{bmatrix} x_c \\ y_c \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

Changing distance $d_a(t)$ from P to A can be described by combining horizontal and vertical component vector equations:

$$d_{x_a}(t) = \begin{bmatrix} x_c \\ 0 \end{bmatrix} - \begin{bmatrix} x_a \\ 0 \end{bmatrix} + t \begin{bmatrix} v_x \\ 0 \end{bmatrix} \text{ and } d_{y_a}(t) = \begin{bmatrix} 0 \\ y_c \end{bmatrix} - \begin{bmatrix} 0 \\ y_a \end{bmatrix} + t \begin{bmatrix} 0 \\ v_y \end{bmatrix}$$

Using PYTHAGORAS we then get:

$$d_a(t) = \sqrt{(x_c - x_a + tv_x)^2 + (y_c - y_a + tv_y)^2}$$

By differentiating this equation by time variable(differential fraction of distance through time equals velocity) we get an equation for velocity:

$$\dot{d}_a(t) = \frac{v_x(x_c - x_a + tv_x) + v_y(y_c - y_a + tv_y)}{\sqrt{(x_c - x_a + tv_x)^2 + (y_c - y_a + tv_y)^2}}$$

By inserting $t = 0$ in the equation we get our measured velocity v_a :

$$\dot{d}_a(t_0) = v_a = \frac{v_x(x_c - x_a) + v_y(y_c - y_a)}{\sqrt{(x_c - x_a)^2 + (y_c - y_a)^2}} = \frac{v_x(x_c - x_a) + v_y(y_c - y_a)}{d_a(0)}$$

Equation for v_b is analogue to this one and by combining the two we get a system of linear equations for v_x and v_y :

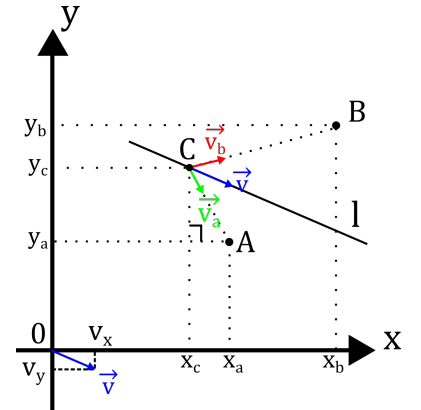
$$\begin{aligned} (x_c - x_a)v_x + (y_c - y_a)v_y &= v_a d_a(0) \\ (x_c - x_b)v_x + (y_c - y_b)v_y &= v_b d_b(0) \end{aligned}$$

Since according to problem text \vec{AC} and \vec{BC} are not parallel this system has a definite solution.

By using CRAMERS³⁵ rule we get:

³⁴This result is derived by using two differentiation facts - formula for differentiating power functions $\frac{dx^a}{dx} = a \cdot x^{a-1}$ and chain rule

³⁵See supplement \rightarrow theory \rightarrow Cramers rule



$$\begin{bmatrix} x_p(t) \\ y_p(t) \end{bmatrix} = \begin{bmatrix} x_c \\ y_c \end{bmatrix} + \frac{t}{(x_c - x_a)(y_c - y_b)(x_c - x_b)(y_c - y_a)} \begin{bmatrix} v_a d_a(0)(y_c - y_a) - v_b d_b(0)(x_c - x_a) \\ v_b d_b(0)(x_c - x_a) - v_a d_a(0)(x_c - x_b) \end{bmatrix}$$

30. Let us define plane \mathcal{P}_{jk} through following properties: it is perpendicular to path between J and K and contains middle point of it. With these it can be described through:

$$(\vec{k} - \vec{j}) \cdot \vec{p} = (\vec{k} - \vec{j}) \cdot \frac{1}{2}(\vec{j} + \vec{k})$$

Vector \vec{p} is an arbitrary point of plane \mathcal{P}_{jk} so we use middle point of path between J and K . Every point of \mathcal{P}_{jk} has the same respective distance to J and K . All points outside of \mathcal{P}_{jk} have other distance from J than from K .

In the same manner we can define plane \mathcal{P}_{kl} :

$$(\vec{l} - \vec{k}) \cdot \vec{p} = (\vec{l} - \vec{k}) \cdot \frac{1}{2}(\vec{k} + \vec{l})$$

Any point of \mathcal{P}_{kl} , has the same respective distance from K and L .

\vec{JK} and \vec{KL} are not parallel therefore planes \mathcal{P}_{jk} and \mathcal{P}_{kl} have a common intersection line l .

Let us choose a random point $R \in l$, then since $R \in \mathcal{P}_{jk}$ we have $|RJ| = |RK|$, and on the other side because $R \in \mathcal{P}_{kl}$ also $|RK| = |RL|$, by combining these two we get the third condition $|RJ| = |RL|$.

Set S we are searching for is therefore a *line*, and more precisely intersection line between \mathcal{P}_{jk} and \mathcal{P}_{kl} .

Its direction vector is perpendicular to normal vectors of both planes. With random point $R_0 \in S$ we then get:

$$S = \{\vec{l}(\tau) = \vec{r}_0 + \tau(\vec{k} - \vec{j}) \times (\vec{l} - \vec{k})\}$$

R_0 can be found as point of intersection between both planes and plane which contains all three points J, K and L .

31. Parallel means that sum $\lambda\vec{a} + \mu\vec{b} + \vec{c}$ is multiple $\nu\vec{d}$ of vector \vec{d} :

$$\lambda\vec{a} + \mu\vec{b} + \vec{c} = \nu\vec{d} \Rightarrow \lambda\vec{a} + \mu\vec{b} - \nu\vec{d} = -\vec{c}.$$

It is a system of linear equations to determine λ, μ and ν , however only λ and μ is of interest for us.

$$\lambda \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix} - \nu \begin{bmatrix} 8 \\ -3 \\ -8 \end{bmatrix} = - \begin{bmatrix} -48 \\ 4 \\ 39 \end{bmatrix}$$

By writing out individual vector component equations we get:

$$2\lambda - 4\mu - 8\nu = 48 \quad (1)$$

$$-\mu + 3\nu = -4 \quad (2)$$

$$5\lambda + 5\mu + 8\nu = -39 \quad (3)$$

We can use equation(2) to substitute $\mu = 4 + 3\nu$ in (1) and (3):

$$2\lambda - 4(4 + 3\nu) - 8\nu = 48 \Rightarrow 2\lambda - 20\nu = 64 \quad (4)$$

$$5\lambda + 5(4 + 3\nu) + 8\nu = -39 \Rightarrow 5\lambda + 23\nu = -59 \quad (5)$$

Now we use equation(4) to substitute $\lambda = 32 + 10\nu$ in (5):

$$5(32 + 10\nu) + 23\nu = -59 \Rightarrow \nu = \frac{-219}{73} = -3 \quad (6)$$

We set $\nu = -3$ into equations $\lambda = 32 + 10\nu$ and $\mu = 4 + 3\nu$ and through it we get answers $\lambda = 2$ and $\mu = -5$.

32. One can substitute forces through other variables to calculate with whole numbers as long as possible which eliminates rounding errors. We substitute force vectors \vec{F}_r and \vec{F}_s through:

$$\vec{F}_r = \lambda \begin{bmatrix} -5 \\ -12 \end{bmatrix} \text{ and } \vec{F}_s = \mu \begin{bmatrix} -7 \\ -3 \end{bmatrix}$$

now we insert them into equilibrium equation:

$$\frac{400N}{\sqrt{12^2 + 7^2}} \begin{bmatrix} 12 \\ 7 \end{bmatrix} + \lambda \begin{bmatrix} -5 \\ -12 \end{bmatrix} + \mu \begin{bmatrix} -7 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To spare writing effort we can substitute $400N/\sqrt{193} = a$, and by writing out vector component equations we get the system:

$$\begin{aligned} 5\lambda + 7\mu &= 12a \cdot 12(-) \\ 12\lambda + 3\mu &= 7a \cdot 5 \end{aligned}$$

By multiplying first equation with 12 and subtracting from it second equation multiplied by 5 we get $69\mu = 109a$.

In the same manner we can multiply first equation by 3 and subtract from it second equation times 7 to get $69\lambda = 13a$.

With it we can calculate magnitude of forces f_r and f_s :

$$f_r = |\vec{F}_r| = |\lambda \begin{bmatrix} -5 \\ -12 \end{bmatrix}| = |\lambda| \cdot \left| \begin{bmatrix} -5 \\ -12 \end{bmatrix} \right| = \frac{13a}{69} \sqrt{5^2 + 12^2} = \frac{13^2 \cdot 400N}{69 \cdot \sqrt{193}} = 70.5N$$

and

$$f_s = |\vec{F}_s| = |\mu \begin{bmatrix} -7 \\ -3 \end{bmatrix}| = |\mu| \cdot \left| \begin{bmatrix} -7 \\ -3 \end{bmatrix} \right| = \frac{109a}{69} \sqrt{7^2 + 3^2} = \frac{109 \cdot 400N \cdot \sqrt{58}}{69 \cdot \sqrt{193}} = 346.4N.$$

33. First one should suppress the wish to calculate \hat{u}_a and \hat{u}_b directly! Giving a bit of thought before starting can save a ton of work!

$$\begin{aligned} \hat{u}_a \times \hat{u}_b &= \left(\frac{1}{\sqrt{7^2 + 2^2 + 4^2}} \begin{bmatrix} 7 \\ 2 \\ 4 \end{bmatrix} \right) \times \left(\frac{1}{\sqrt{5^2 + 1^2 + 3^2}} \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix} \right) = \\ &= \frac{1}{\sqrt{69 \cdot 35}} \left(\begin{bmatrix} 7 \\ 2 \\ 4 \end{bmatrix} \times \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix} \right) = \frac{1}{\sqrt{2415}} \begin{bmatrix} 10 \\ -41 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.2035 \\ -0.8343 \\ 0.0610 \end{bmatrix} \end{aligned}$$

34. Because it is unreasonable. Vector \vec{v} creates 0 degree angle with itself- therefore- because of the definition of cross product it is always $\vec{v} \times \vec{v} = 0$. Besides that - \vec{v}^2 is actually common, but used as $\vec{v} \cdot \vec{v} = |\vec{v}|^2$.

35. Those who approach the problem carefully will easily realize that one of the factors appears twice - once on the right and once on the left side of the sum. One can not simply pull it out because cross-product is more sensitive than *Mimosa pudica* when it comes to changing sides of the factors. Because of that one could calculate like this:

$$\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \times \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \times \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \times \left(\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$$

36. Value of dot product of two unit vectors is never higher than 1 since:

$$\hat{u}_v \cdot \hat{u}_w = |\hat{u}_v| \cdot |\hat{u}_w| \cdot \cos \angle(\hat{u}_v, \hat{u}_w) = 1 \cdot 1 \cdot \cos \angle(\hat{u}_v, \hat{u}_w) = \cos \angle(\hat{u}_v, \hat{u}_w) \leq 1$$

and cosine has the value of 1 only if the argument is 0, therefore the angle between both vectors is 0.

Both vectors are therefore parallel and point in the same direction. They are a positive multiple of each other: $\vec{v} = \lambda \vec{w}$.

We can calculate factor λ from x-component: $2 = \lambda \cdot 6 \Rightarrow \lambda = \frac{1}{3}$.

With it $y = \frac{1}{3} \cdot (-4) = -\frac{4}{3}$ and $z = \frac{1}{3} \cdot (-5) = -\frac{5}{3}$.

37. Finally a simple problem - one can start calculation without further a due!

Let given force be $\vec{F}_1 = [a, b]^T$, and the one we are searching for $\vec{F}_2 = [c, d]^T$. We set them as 2D-vectors by using the fact that they start from the same point-which means that they are in the same plane.

We can set the components in length equation: $\left| \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \right| = \left| \begin{bmatrix} a \\ b \end{bmatrix} \right| \Rightarrow (a+c)^2 + (b+d)^2 = a^2 + b^2$, and length of \vec{F}_2 gets us $c^2 + d^2 = F_2^2$. we can simplify the first equation by expanding it: $(a+c)^2 + (b+d)^2 = a^2 + 2ac + c^2 + b^2 + 2bd + d^2 = a^2 + b^2 \Rightarrow 2ac + 2bd + \underbrace{c^2 + d^2}_{F_2^2} = 0$. With it we have two equations for two

unknown variables: $2ac + 2bd + F_2^2 = 0$ and $c^2 + d^2 = F_2^2$.

Unfortunately this system is non-linear, so we apply classical method of expressing one variable from one equation and then substitute it with this expression in second equation. We could use first(linear) equation to do that but different cases must be considered if $a = 0$ or $a \neq 0$.

One could give it a second thought if there is a better way to solve the problem. There is no further information

then the fact of the existence of two forces. There is no fixed coordinate-system either, which means that we can apply one by ourselves - it would be reasonable to choose one that simplifies our problem. It is harmless for our task since we are searching for angle between the two vectors and not their coordinate values. We set x -axis so that it has the same direction as \vec{F}_1 . Its length is given, if one thinks about it all one needs for the solution of this problem are length values F_1 and F_2 of both vectors. We have $\vec{F}_1 = [F_1, 0]^T$ with $F_1 > 0$. We set F_1 in place of a and 0 in place of b in our equation:

$$2F_1c + 0 \cdot d + F_2^2 = 0 \Rightarrow c = -F_2^2/2F_1 \text{ and } c^2 + d^2 = F_2^2$$

It turns out that second equation is not needed because of lucky selection of coordinate-system. Let us proceed with calculation of the angle α between both vectors by using properties of the dot-product :

$$\begin{aligned} \vec{F}_1 \cdot \vec{F}_2 &= |\vec{F}_1| \cdot |\vec{F}_2| \cdot \cos \alpha = \begin{bmatrix} F_1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = F_1c + 0 \cdot d = \frac{F_1 F_2^2}{2 F_1} = -\frac{F_2^2}{2} \\ \Rightarrow \cos \alpha &= -\frac{F_2^2/2}{2F_1 F_2} = -\frac{F_2}{2F_1} \Rightarrow \alpha = \pi - \arccos \frac{F_2}{2F_1} \end{aligned}$$

The result is definitely an acute angle.

For problem to have a valid solution it is necessary that $F_2 \leq 2F_1$.

38. Basic idea: if a triangle is equilateral all of its inner angles have the width of 60° .

Let substitute point $S = \vec{l}(t_s) = [1, -1, 4]^T + t_s[2, 3, -1]^T$ be any of points Q or R . In this case path SP must have 60° angle with direction vector of l . By using the definition of dot product we can calculate:

$$\begin{aligned} \left[\begin{bmatrix} 7 \\ 6 \\ 15 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + t_s \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right) \right] \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} &= \begin{bmatrix} 6 - 2t_s \\ 7 - 3t_s \\ 11 + t_s \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \\ \sqrt{(6 - 2t_s)^2 + (7 - 3t_s)^2 + (11 + t_s)^2} \cdot \sqrt{4 + 9 + 1} \cdot \cos 60^\circ & \\ 22 - 14t_s &= \frac{1}{2} \cdot \sqrt{206 - 44t_s + 14t_s^2} \cdot 14 \end{aligned}$$

By squaring both sides of the last equation we get:

$$484 - 616t_s + 196t_s^2 = \frac{14}{4}(206 - 44t_s + 14t_s^2) \text{ or simplified: } 147t_s^2 - 462t_s - 237 = 0.$$

This quadratic equation has two solutions that correspond to parameters for points Q and R . They are $t_{s1} = 3.592$ and $t_{s2} = -0.4489$, therefore $Q = (0.102, -2.347, 4.449)^T$ and $R = (8.183, 9.775, 0.408)^T$.

39. Area of a triangle equals to one half of its baseline times its height. Height in this case is the distance from P to l . Let t_0 be the parameter value for point on line l , which connected with P makes a path perpendicular to l . In this case vector from P to $\vec{l}(t_0)$ in P is perpendicular to direction vector of l :

$$\left[\begin{bmatrix} 10 \\ 6 \\ 15 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + t_0 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right) \right] \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 - 2t_0 \\ 7 - 3t_0 \\ 11 + t_0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = 28 - 14t_0 = 0.$$

We therefore get $t_0 = 2$ and by inserting it into the line equation we get $\vec{l}(t_0) = [5, 5, 2]^T$. Length of the difference vector $\vec{P} - \vec{l}(t_0)$ equals to the height of our triangle $h = \sqrt{(7-5)^2 + (6-5)^2 + (15-2)^2} = \sqrt{174}$. Length of the baseline l_b can be calculated by using area and height values:

$$\frac{1}{2} \cdot l_b \cdot \sqrt{174} = 109 \Rightarrow l_b = 218/\sqrt{174}.$$

We can calculate coordinates of point R by adding a path of this length to $\vec{l}(t_0)$ in both directions of line l . To do that we multiply baseline length value with unit vector of lines direction vector and add or subtract it from $\vec{l}(t_0)$:

$$R_{1,2} = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix} \pm \frac{218}{\sqrt{174} \cdot \sqrt{2^2 + 3^2 + 1^2}} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix} \pm 4.4169 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

Result: $R_1 = [13.83, 18.25, -2.42]^T$ and $R_2 = [-3, 83, -8.25, 6.42]^T$.

40. Connection line between starting points of both lines must be in this plane, therefore $[7, 0, 3]^T - [1, 2, 3]^T = [6, -2, 0]^T$ is parallel to the plane and its cross-product with direction vector of the first line results in a normal vector of this plane: $[6, -2, 0]^T \times [2, 5, -1]^T = [-2, -6, -34]^T$. This normal must also be perpendicular to

the direction vector of l_2 and by using dot product we can calculate the value of a : $[1, 3, 17]^T \cdot [6, a, 4]^T = 6 + 3a + 68 = 74 + 3a = 0 \Rightarrow a = -74/3$.

41. First we search for unit vectors \hat{i} and \hat{j} of the inner coordinate system of \mathcal{P} expressed in space coordinates. (One odd to contemplate in this situation: any given vector objectively exists. Its coordinates depend on the choice of coordinate system - a subjective human choice. Use of another coordinate system changes the representation of the vector but not the vector itself.)

Besides: it is nowhere mentioned that \hat{i} and \hat{j} must have a length of 1 in space; internal length unit of plane can differ from the space one. They even do not need to be of the same length in space coordinates! ³⁶

Let $\vec{i} = [x_i, y_i, z_i]^T$ and $\vec{j} = [x_j, y_j, z_j]^T$ be the representation of \hat{i} and \hat{j} in space coordinates. What do we know about them?

They are parallel to the plane \mathcal{P} and it is parallel to \vec{AB} and \vec{AC} . Last two vectors are not parallel to each other and are therefore linearly independent.³⁷ Vectors \vec{AB}, \vec{AC} and \vec{i} are in the same plane with origin and thus linearly dependent. Vector \vec{i} can be therefore get constructed through linear combination of \vec{AB} and \vec{AC} .

We assume that $i = \lambda_i \vec{AB} + \mu_i \vec{AC}$ and $j = \lambda_j \vec{AB} + \mu_j \vec{AC}$. All there is left to calculate are 4 coefficients (Only four! If we would have chosen to work with coordinates of \vec{i} and \vec{j} it would be 6 unknown values).

In S -coordinates $\vec{AB} = \vec{B} - \vec{A} = [4, -2]^T = 4\hat{i} - 2\hat{j}$ and $\vec{AC} = [6, -1]^T = 6\hat{i} - \hat{j}$.

By inserting these values in our assumption equation we get $\vec{AB} = 4(\lambda_i \vec{AB} + \mu_i \vec{AC}) - 2(\lambda_j \vec{AB} + \mu_j \vec{AC}) = (4\lambda_i - 2\lambda_j)\vec{AB} + (4\mu_i - 2\mu_j)\vec{AC}$; right hand side of the equation indicates that $4\lambda_i - 2\lambda_j = 1$ and $4\mu_i - 2\mu_j = 0$.

We can use the same method on $\vec{AC} = 6(\lambda_i \vec{AB} + \mu_i \vec{AC}) - (\lambda_j \vec{AB} + \mu_j \vec{AC}) = (6\lambda_i - \lambda_j)\vec{AB} + (6\mu_i - \mu_j)\vec{AC}$ from which follows that $6\lambda_i - \lambda_j = 0$ and $6\mu_i - \mu_j = 1$.

We have two systems of linear equations with two variables in each:

$$\begin{aligned} 4\lambda_i - 2\lambda_j &= 1 & 4\mu_i - 2\mu_j &= 0 \\ 6\lambda_i - \lambda_j &= 0 & 6\mu_i - \mu_j &= 1. \end{aligned}$$

Their solutions are $\lambda_i = -1/8, \mu_i = 1/4$ and $\lambda_j = -3/4, \mu_j = 1/2$.

With these values in space coordinates: $\vec{i} = -\frac{1}{8} \begin{bmatrix} -8 \\ -4 \\ 6 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} -8 \\ -6 \\ 11 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ and $\vec{j} = -\frac{3}{4} \begin{bmatrix} -8 \\ -4 \\ 6 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -8 \\ -6 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

These vectors need to be orthogonal for a construction of coordinate system, hence their dot product must equal 0. This is indeed the case.

a) It is:

$$\vec{0}_{\mathcal{P}} = \vec{A} - \vec{A} = \vec{A} + 2\vec{i} - 3\vec{j} = \begin{bmatrix} 12 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$$

b) Approach:

$$\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \lambda = 1, \mu = 0$$

Therefore coordinates of the point are $[1, 0]^T$.

c)

$$\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} + 5 \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} - 6 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -13 \\ -6 \\ 7 \end{bmatrix}$$

42. Fast answer: $\vec{y} = \lambda \vec{x}$ must be true. Or $\vec{x} = \lambda \vec{y}$ if $\vec{x} = \vec{0}$.³⁸

One can not simply test **if** " $y_1/x_1 = y_2/x_2$ **and** $y_1/x_1 = y_3/x_3$ ", since one of the 3 fraction denominators might be 0. One could exclude these cases with a test but that would only make the logical structure even more complicated.

Both equations are proportions and can be also written as $x_1 \cdot y_2 = x_2 \cdot y_1$ and $x_1 \cdot y_3 = x_3 \cdot y_1$.

One could therefore write a program that tests if $x_1 \cdot y_2 - x_2 \cdot y_1$ and $x_1 \cdot y_3 - x_3 \cdot y_1$ are equal to zero. If that

³⁶Let us assume that we are measuring the distance between two points through time that light ray needs to travel from one point to the other. B and C are both 1 cm away from A - therefore geometrically in the same distance from A . Medium between A and B is glass ($n > 1$), but between A and C air ($n = 1$). With it if we measure with travel time of light B is further away from A than C .

³⁷See Supplement \rightarrow Theory \rightarrow linear dependency

³⁸This is a common abbreviation sign for null-vector; a vector whose all components are 0 as in $[0, 0, 0, \dots, 0, 0]^T$

is the case than vectors are parallel. In this case $x_3 \cdot y_2 - x_2 \cdot y_3 = 0$ is also valid.

Geometrically this means that cross-product disappears which is also a criteria that can be used to prove that 2 vectors are parallel.

If any of the vectors is equal to Θ , then they are seen as parallel too.

If one wants to know if they point in the same direction then one can test for $\lambda > 0$; if parallel-test succeeded then $x_1y_1 \geq 0$ and $x_2y_2 \geq 0$ and $x_3y_3 \geq 0$ must be true.

Why should components of vectors be integer numbers? - see supplement question B at the end of this document.

$$43. |\vec{r} - \vec{a}| = |\vec{r} - \vec{b}| \Leftrightarrow |\vec{r} - \vec{a}|^2 = |\vec{r} - \vec{b}|^2 \Leftrightarrow (\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{a}) = (\vec{r} - \vec{b}) \cdot (\vec{r} - \vec{b}) \Leftrightarrow \vec{r} \cdot \vec{r} - 2\vec{r} \cdot \vec{a} + \vec{a} \cdot \vec{a} = \vec{r} \cdot \vec{r} - 2\vec{r} \cdot \vec{b} + \vec{b} \cdot \vec{b} \Leftrightarrow 2\vec{r} \cdot \vec{b} - 2\vec{r} \cdot \vec{a} + \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{b} = 0 \Leftrightarrow 2\vec{r} \cdot (\vec{b} - \vec{a}) + (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0 \Leftrightarrow [\vec{r} - \frac{1}{2}(\vec{a} + \vec{b})] \cdot (\vec{a} - \vec{b}) = 0$$

Last equation is an equation of a plane in point normal form. Points of this plane are represented by \vec{r} .

Point that fixes this plane is $\frac{1}{2}(\vec{a} + \vec{b})$ - point in the middle of path between \vec{a} and \vec{b} . Normal vector of the plane is $\vec{a} - \vec{b}$ - direction vector of the path between \vec{a} and \vec{b} . Everything as expected.

44. Shadow is the intersection line between given plane and plane that is created through points L, P and Q . This line is parallel to both planes and therefore perpendicular to their normal vectors. Lines direction vector \vec{d} can therefore get calculated by using cross-product of these normal vectors. One of them is clear from standard equation of the plane and other can be calculated through:

$$\vec{LP} \times \vec{LQ} = \begin{bmatrix} -4 \\ -5 \\ -21 \end{bmatrix} \times \begin{bmatrix} 2 \\ -6 \\ -19 \end{bmatrix} = \begin{bmatrix} -31 \\ -118 \\ 34 \end{bmatrix}$$

and one gets

$$\vec{d} = \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} \times \begin{bmatrix} -31 \\ -118 \\ 34 \end{bmatrix} = \begin{bmatrix} -742 \\ 220 \\ 87 \end{bmatrix}$$

87 is product of 3 and 29 and neither 742 nor 220 can be divided by these; therefore this vector cannot get simplified.

Shadow is not the whole intersection line between the 2 planes, it is only on the side where line running through P and Q seems to be in front of plane $x + y + 6z = -60$ when observed from L . We need to find coordinates of intersection point of line and plane. To do that we set components of the line equation into plane equation:

$$\vec{l}_{PQ}(\lambda) = \vec{P} + \lambda(\vec{Q} - \vec{P}) = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 4 - (-2) \\ -1 - 0 \\ 3 - 1 \end{bmatrix} \\ (-2 + 6\lambda) + (0 - \lambda) + 6(1 + 2\lambda) = 4 + 17\lambda = -60 \Rightarrow \lambda = -64/17$$

With it the coordinates of intersection point are:

$$\vec{P}_i = \begin{bmatrix} 2 + 6 \cdot (-64/17) \\ 0 - (-64/17) \\ 1 + 2 \cdot (-64/17) \end{bmatrix} = \begin{bmatrix} -418/17 \\ 64/17 \\ -111/17 \end{bmatrix}$$

Line that contains the shadow can be then described as:

$$\vec{l}_{sh}(\mu) = \frac{1}{17} \begin{bmatrix} -418 \\ 64 \\ -111 \end{bmatrix} + \mu \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix}$$

Shadow begins at $\mu = 0$. Left to find out is whether it is on the side of $\mu < 0$ or $\mu > 0$.

To do that we need to research if L is above or under the plane if we choose z -axis pointing upwards. To do that we can insert x and y values of L in plane equation to find the corresponding points (of the plane) z -coordinate and compare it to the one of L :

$$2 + 5 + 6z = -60 \Rightarrow z < 0, \text{ however } z_L = 22$$

Therefore the plane lies under L .

Now we choose a positive value for μ and compare the height of line value it is creating with corresponding height of the plane as in previous calculation. If $\mu = 1/17 > 0$, then $\vec{l}_{sh}(1/17) = \frac{1}{17}[-412, 63, -109]^T$; we set x - and y - values in the plane equation:

$$\frac{-412}{17} + \frac{63}{17} + 6z = -60 \Rightarrow z = \frac{-(671/6)}{17}$$

Since $-109 > -671/6$ we can conclude that positive values of μ provide points of the line between P and Q that are above the plane and therefore throw a shadow on it.

Result: ray of shadow can be described through:

$$\vec{l}_{sh}(\mu) = \frac{1}{17} \begin{bmatrix} -418 \\ 64 \\ -111 \end{bmatrix} + \mu \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix}, \quad \mu \geq 0.$$

45. With Q we already have a start point for the horizontal line, we only need to find its direction vector. Since it is horizontal its z -coordinate must be equal to zero: $\vec{d} = [x_d, y_d, 0]^T$. Since this line lies in \mathcal{P} , it and its direction vector must be orthogonal to normal vector $\vec{n}_{\mathcal{P}} = [2, -5, -2]^T$ of the plane and dot product of both is therefore 0: $2x_d - 5y_d - 2 \cdot 0 = 0$, we can for example choose $x_d = 5$ and $y_d = 2$.

Result:

$$\vec{l}(\lambda) = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix} + \lambda \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$$

A *vertical* line $\in \mathcal{P}$ through Q does not exist.

How come?

One might think that *horizontal* and *vertical* are just opposites of the same matter but that is not the case. Word *horizontal* contains two lineary independent directions, *vertical* however just a single one. Set of all points that are directly over or under Q create a line, set of points that are in the same height against Q create a plane. And this plane has an intersection line with plane \mathcal{P} , while the vertical line through Q is (with high probability) not running in \mathcal{P} .

If this vertical line would run in \mathcal{P} , then \mathcal{P} should be a vertical surface and have a horizontal normal vector. Normal vectors z -component is however -2 and not 0 .

46. Since $P(x)$ is exactly over x -axis its y -coordinate must also equal zero: $P(x) = (x, 0, z)$. We can set its components into plane equation: $14x + 311z = -1072$ and get $z = -\frac{14}{311}x + \dots$. It is enough to provide an answer to our question - since coefficient before x is negative the value of resulting linear function decreases with increasing x and therefore point $P(x)$ falls.

47. First we need to choose a coordinate system. It can be done arbitrary and according to our wish to make calculations as simple as possible. We set x -axis in the same direction as first force. We only need $2D$ -system so we choose the plane that is span up by these two force vectors.

Forces are not parallel; if they would be, their sum should equal $80N$ or $0N$.

First force is (according to our chosen coordinate system and N sign excluded) $\vec{F}_1 = [40, 0]^T$, with angle α between the two vectors we get $\vec{F}_2 = [40 \cos \alpha, 40 \sin \alpha]^T$.

Using magnitude(length) of resulting vector we get:

$$70 = \sqrt{(40 + 40 \cos \alpha)^2 + 40^2 \sin^2 \alpha} \Rightarrow 4900 = 40^2(1 + 2 \cos \alpha + \underbrace{\cos^2 \alpha + \sin^2 \alpha}_1) = 1600(2 + 2 \cos \alpha)$$

or

$$\cos \alpha = \frac{4900}{3200} - 1 \Rightarrow \alpha = \pm 57.9^\circ.$$

One can set angle sign as positive since in this case its direction is irrelevant.

48. a) If l_1 and l_2 are parallel we can assign them the same direction vector $\vec{d} = [a, b, c]^T$. We use it and assign parameters p and q to describe lines l_1 and l_2 :

$$\vec{l}_1(p) = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} + p \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \vec{l}_2(q) = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + q \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

With certain values of p and q as well as s and t we can calculate coordinates of intersection points with lines l_1^* and l_2^* :

$$\begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} + p \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + q \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

Both equations split into their components result into 6 equations for 7 unknown coefficients; that would seem as too few however length of direction vector $[a, b, c]^T$ can vary so it is not a problem.

$$\begin{array}{ccc}
2 + pa = -2 + t & 1 + qa = 3 + 2s & 4 + pa = t & -2 + qa = 2s \\
6 + pb = 1 + t & qb = 3 + s & 5 + pb = t & -3 + qb = s \\
1 + pc = 6 - 3t & 3 + qc = -2 + 5s & -5 + pc = -3t & 5 + qc = 5s
\end{array} \Rightarrow$$

From the resulting 6 equations we use left top one $t = 4 + pa$ and right middle one $s = qb - 3$ to substitute t and s in rest of the equations:

$$\begin{array}{ccc}
5 + pb = 4 + pa & p(a - b) = 1 & -2 + qa = 2qb - 6 \Rightarrow q(a - 2b) = -4 \\
-5 + pc = -12 - 3pa \Rightarrow p(3a + c) = -7 & & 5 + qc = 5qb - 15 \Rightarrow q(c - 5b) = -20
\end{array}$$

Resulting product equations bring a value that is not zero - therefore we can divide them by one of the products terms, as for example:

$$\begin{array}{l}
\frac{p(a - b)}{(a - b)} = \frac{1}{(a - b)} \Rightarrow p = \frac{1}{(a - b)} \\
\frac{q(a - 2b)}{(a - 2b)} = \frac{-4}{(a - 2b)} \Rightarrow q = \frac{-4}{(a - 2b)}
\end{array}$$

Using this we substitute p and q in two remaining equations:

$$\begin{array}{l}
\frac{1}{(a - b)} \cdot (3a + c) = 7 \Rightarrow 10a - 7b + c = 0 \\
\frac{-4}{(a - 2b)} \cdot (c - 5b) = -20 \Rightarrow 5a - b - c = 0
\end{array}$$

Equations on the right side added to each other eliminate variable c : $15a - 12b = 0$. We can set either a or b to an arbitrary value to calculate the other. Simple whole number version would be $a = 4$ and $b = 5$, in this case $c = -5$.

Result:

$$l_1 : \vec{l}_1(p) = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} + p \begin{bmatrix} 4 \\ 5 \\ -5 \end{bmatrix} \quad l_2 : \vec{l}_2(q) = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + q \begin{bmatrix} 4 \\ 5 \\ -5 \end{bmatrix}$$

b) by setting P_1 and P_2 equal to line equations l_1^* and l_2^* one can easily see that the resulting equation systems have no solutions and therefore these points are (correspondingly) not on lines l_1^* and l_2^* .

Let us assume t_i and therefore $\vec{P}_1^*(t_i)$ as given; in this case we have a clear definition of line l'_1 through points P_1 and $P_1^*(t_i)$.

We know that a plane that contains both line l_2^* and point P_2 exists. It intersects with line l'_1 at point P_{i1} . It is possible to calculate this point. Line l'_2 running through P_{i1} and P_2 also lies in this plane and intersects with l_2^* (in the highly probable case that these lines are not parallel to each other). This intersection point is the one that we are searching for - $\vec{P}_2^*(s_i)$.

Let us first find an equation for the mentioned plane in standard form. One needs two non-parallel vectors to calculate a normal vector of this plane. We use direction vector of l_2^* and vector running from P_2 to starting point of l_2^* and calculate their cross-product (which results in a normal-vector of the plane):

$$\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \times \begin{bmatrix} 3 - 1 \\ 3 - 0 \\ -2 - 3 \end{bmatrix} = \begin{bmatrix} -20 \\ 20 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} -20 \\ 20 \\ 4 \end{bmatrix}$$

To calculate standard equation we use P_2 as point in the plane³⁹

$$\begin{bmatrix} -20 \\ 20 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -20 \\ 20 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \Rightarrow -5x + 5y + z = -2$$

. As mentioned line l'_1 runs through points P_1 and $P_1^*(t_i)$; we assign its line equation with a parameter p' :

$$\vec{l}_1(p') = \vec{P}_1 + p'(\underbrace{\vec{P}_1^*(t_i) - \vec{P}_1}_{=\vec{l}_1^*(t_i)}) = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} + p' \left(\begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix} + t_i \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} \right)$$

To calculate intersection point of l'_1 with plane we set lines equation components into standard equation of the plane:

³⁹for elaborate explanation see solution of question l on page 2 of this document

$$-5[2 + p'(-2 + t_i - 2)] + 5[6 + p'(1 + t_i - 6)] + [1 + p'(6 - 3t_i - 1)] = 21 - 3p't_i = -2 \Rightarrow p' = 23/3t_i$$

p' is undefined if $t_i = 0$. In this case direction vector of l'_1 would be equal to $[-2, 1, 6]^T - [2, 6, 1]^T = [-4, -5, 5]^T$ and would therefore be orthogonal (their dot product equals 0) to normal vector of the plane $[-5, 5, 1]^T$; which would mean that l'_1 is parallel to it and there is no point of intersection.

We can now use value of p' in line equation of l'_1 to get a formula for intersection point P_{i1} :

$$P_{i1}(t_i) = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} + \frac{23}{3t_i} \begin{bmatrix} -4 + t_i \\ -5 + t_i \\ 5 - 3t_i \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 29 \\ 41 \\ -66 \end{bmatrix} + \frac{23}{3t_i} \begin{bmatrix} -4 \\ -5 \\ 5 \end{bmatrix}$$

We are now able to write an equation for line l'_2 :

$$\vec{l}_2(q) = \vec{P}_2 + q(\vec{P}_{i1} - \vec{P}_2) = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + q \left(\frac{1}{3} \begin{bmatrix} 29 \\ 41 \\ -66 \end{bmatrix} + \frac{23}{3t_i} \begin{bmatrix} -4 \\ -5 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + q \left(\frac{1}{3} \begin{bmatrix} 26 \\ 41 \\ -75 \end{bmatrix} + \frac{23}{3t_i} \begin{bmatrix} -4 \\ -5 \\ 5 \end{bmatrix} \right)$$

Now all that is left is intersection between lines l'_2 and l_2^* :

$$\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + q \left(\frac{1}{3} \begin{bmatrix} 26 \\ 41 \\ -75 \end{bmatrix} + \frac{23}{3t_i} \begin{bmatrix} -4 \\ -5 \\ 5 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix} + s_i \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \quad | \cdot 3t_i$$

after multiplying both sides by $3t_i$ we extract x- and y- component equations to find a relation for parameters q and s_i :

$$\begin{aligned} 3t_i + 26qt_i - 92q &= 9t_i + 6s_it_i \\ 41qt_i - 115q &= 9t_i + 3s_it_i \quad | \cdot 2(-) \end{aligned}$$

After subtraction of second equation times two from first equation we got $-56qt_i + 138 = -12t_i$ left. We can express $q = 12t_i/(56t_i - 138) = 6t_i/(28t_i - 69)$ explicitly and substitute it in second equation of previous equation system:

$$41 \cdot \frac{6t_i}{28t_i - 69} \cdot t_i - 115 \cdot \frac{6t_i}{28t_i - 69} = 9t_i + 3s_it_i \quad | \cdot (28t_i - 69)$$

After multiplying both sides by $28t_i - 69$ we get:

$$\begin{aligned} 41 \cdot 6t_i^2 - 115 \cdot 6t_i &= 9t_i \cdot (28t_i - 69) + 3s_it_i \cdot (28t_i - 69) \\ \Rightarrow 246t_i^2 - 690t_i &= 252t_i^2 - 621t_i + 84s_it_i^2 - 207s_it_i - 6t_i^2 - 69t_i = s_i(84t_i^2 - 207t_i) \\ \Rightarrow s_i &= \frac{246t_i^2 - 690t_i}{84t_i^2 - 207t_i} = \frac{82t_i - 230}{28t_i - 69} \end{aligned}$$

We can now substitute s in line equation of l_2^* to get our answer of expressing $\vec{P}_2^*(s_i)$ through $\vec{P}_1^*(t_i)$:

$$\vec{P}_2^*(s_i[t_i]) = \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix} + \frac{82t_i - 230}{28t_i - 69} \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

49. If we know vectors \vec{a} components, we also know its length a . Since $\vec{a} = \vec{b} + \vec{c}$ it must also be true that $|\vec{a}| = |\vec{b} + \vec{c}|$. By using following property of the dot product of any given vector \vec{x} : $|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}}$ we can transform (by squaring both sides) previous equation into following:

$$\vec{a} \cdot \vec{a} = (\vec{b} + \vec{c}) \cdot (\vec{b} + \vec{c}) = \vec{b} \cdot \vec{b} + 2\vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{c} = b^2 + 2bc \cdot \cos \alpha + c^2 = a^2$$

Since $b > 0$ and $c > 0$ we get following equation for cosine of the angle:

$$\cos \alpha = \frac{a^2 - b^2 - c^2}{2bc}$$

This equation is only valid for values between -1 and $+1$. For example it is impossible to split a vector of length 7 in sum of vectors of lengths 4 and 2.

We can choose a random vector \vec{b} of a given length and then calculate $\vec{c} = \vec{a} - \vec{b}$. This problem has therefore no one definite answer.

50. We assign parameter λ_Q as value that gets us point Q when inserted into line equation of l . Now we can calculate $\vec{P}\vec{Q}$:

$$\vec{PQ} = \vec{Q} - \vec{P} = [7, 2, 9]^T + \lambda_Q[-1, -1, 5]^T - [2, 1, 4]^T = [5, 1, 5]^T + \lambda_Q[-1, -1, 5]^T$$

Since it has 30° rise against horizontal, angle between it and unit vector of z -axis $\hat{z} = [0, 0, 1]^T$ must be 60° . By using properties of the dot product we get:

$$\begin{bmatrix} 5 - \lambda_Q \\ 1 - \lambda_Q \\ 5 + 5\lambda_Q \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \left| \begin{bmatrix} 5 - \lambda_Q \\ 1 - \lambda_Q \\ 5 + 5\lambda_Q \end{bmatrix} \right| \cdot \left| \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right| \cdot \cos 60^\circ$$

we can expand this to:

$$5 + 5\lambda_Q = \underbrace{\frac{1}{2}}_{=\cos 60^\circ} \sqrt{(5 - \lambda_Q)^2 + (1 - \lambda_Q)^2 + (5 + 5\lambda_Q)^2}$$

Through division of left side by the fraction on the right and pulling out a factor from the resulting expression we get:

$$10(1 + \lambda_Q) = \sqrt{(5 - \lambda_Q)^2 + (1 - \lambda_Q)^2 + (5 + 5\lambda_Q)^2}$$

we can now square both sides to get rid of the square-root:

$$\begin{aligned} 100(1 + 2\lambda_Q + \lambda_Q^2) &= (5 - \lambda_Q)^2 + (1 - \lambda_Q)^2 + (5 + 5\lambda_Q)^2 \\ 25 - 10\lambda_Q + \lambda_Q^2 + 1 - 2\lambda_Q + \lambda_Q^2 + 25 + 50\lambda_Q + 25\lambda_Q^2 &= 51 + 38\lambda_Q + 27\lambda_Q^2 \\ 73\lambda_Q^2 + 162\lambda_Q + 49 &= 0 \\ \lambda_Q^2 + \frac{162}{73}\lambda_Q + \frac{49}{73} &= 0 \\ \lambda_{Q_{1,2}} &= \frac{-81 \pm \sqrt{81^2 - 49 \cdot 73}}{73} = \frac{-81 \pm \sqrt{2984}}{73} = \frac{-81 \pm 2\sqrt{746}}{73} \end{aligned}$$

It is a fact(from computer science) that $32^2 = 1024 > 1000$, therefore $32 > \sqrt{1000} > \sqrt{746}$ and $81 > 64 > 2\sqrt{746}$ Both zeros of the quadratic equation are thus negative.

z -coordinate of the direction vector of l is positive, $\vec{l}(\lambda)$ will therefore point more upwards with rising value of λ . Our calculation brought two values. How can we interpret this ?. Well, path between P to Q both falls and rises depending from the viewpoint. We want to know the case where it rises and therefore has the higher value of λ_Q :

$$\lambda_Q = \frac{2\sqrt{746} - 81}{73} = -1.85789 \Rightarrow Q = \begin{pmatrix} 6.85789 \\ 2.85789 \\ -4.28945 \end{pmatrix}$$

51. There are two planes that meet this condition. We will search for their normal-form equation, so we need to find the corresponding normal vectors. They must be perpendicular to the vector between both points: $[1, 4, 1]^T - [2, 2, 2]^T = [-1, 2, -1]^T$.

Since the plane we are searching for is definitely not vertical its z -coordinate cannot equal zero. We can therefore arbitrary set its value to 1: $\vec{n} = [a, b, 1]^T$.

From its orthogonality with vector between points we get: $-a + 2b - 1 = 0$ or $a = 2b - 1$. Since \mathcal{P} has 45° rise its normal vector will have the same angle against vertical unit vector $\hat{u}_z = [0, 0, 1]^T$. From dot product between them we get:

$$\begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \left| \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \right| \cdot 1 \cdot \cos 45^\circ$$

by expanding the equation we get:

$$1 = \sqrt{a^2 + b^2 + 1} \frac{\sqrt{2}}{2} = \sqrt{a^2 + b^2 + 1} / \sqrt{2}$$

Squaring both sides provides us following equation with corresponding two solutions:

$$2 = a^2 + b^2 + 1 \Rightarrow 1 = (2b - 1)^2 + b^2 \Rightarrow 0 = 5b^2 - 4b = 5b(b - 0.8)$$

As result we get two normal vectors $n_1 = [-1, 0, 1]^T$ and $n_2 = [0.6, 0.8, 1]^T$. We perform dot product with $[1, 4, 1]^T$ and get two equations for plane \mathcal{P} :

$$z - x = 0 \quad \text{and} \quad 0.6x + 0.8y + z = 4.8$$

Supplement

Problems:

A.)Function $f(x)$ is stable⁴⁰ and defined in the complete set of real numbers. $f(x_1 + x_2) = f(x_1) + f(x_2)$ is valid for all x_1 and x_2 . Which form does function $f(x)$ have?

B.)Two 3D-vectors $\vec{x} = [x_1, x_2, x_3]^T$ and $y = [y_1, y_2, y_3]^T$ are given. Their components x_k and y_k are real type numbers.

Write a function which decides if these vectors are parallel(same direction is not necessary) or not!

⁴⁰see supplement→ theory→ stability

Solutions:

A.) Most students would prefer more tangible questions, for example: What is the area of following pentagon? When ingenious solution does not come to ones mind on its own then it is time to gather facts and hope that earlier or later one will stumble upon some rules. Chances for this to occur might increase if by doing so we keep our eyes open.

Let us start with for this particular case somewhat interesting circumstance : It is common knowledge (at least in mirrored form since we know that $a = b \Rightarrow b = a$) that $0 = 0 + 0$. We use this fact in the given equation and get:

$$f(0) = f(0 + 0) = f(0) + f(0) \Rightarrow 0 = f(0)$$

Logical conclusion on right side results by subtracting $f(0)$ from both sides of equation to the left. Our function of interest therefore intersects with y-axis at point zero.

Furthermore: what happens if there are three variables as argument of the function? Nothing in this regard is mentioned in the problem text. This case can get reduced to two variables through combining two of the three into one - we use an extra pair of braces to apply this:

$$f(x_1 + x_2 + x_3) = f((x_1 + [x_2 + x_3])) = f(x_1) + f(x_2 + x_3) = f(x_1) + [f(x_2) + f(x_3)] = f(x_1) + f(x_2) + f(x_3)$$

It is easy to see that four, five and more arguments would provide equations analogue to this one.

Let us now assume that we have n arguments that all have the *same* value x :

$$f(n \cdot x) = f(x + x + \dots + x) = f(x) + f(x) + \dots + f(x) = n \cdot f(x)$$

Natural factor n can be therefore pulled out of function argument. The fact that n is a natural number is important because previous proof is based on primary definition of multiplication which is repeated addition of the same number.

Is the same operation possible for non-natural numbers?

Lets proceed in systematic fashion and check negative integers as first. While at it, it would be interesting to find out something about $f(-x)$: how does it compare to $f(x)$ - and is there any connection between the two of them?

It is not only $0 = 0 + 0$, but also $0 = x - x$ (better known while being read from right to left), for all numbers x . With it:

$$0 = f(0) = f(x - x) = f(x + [-x]) = f(x) + f(-x) \Rightarrow f(-x) = -f(x)$$

Logical conclusion on right side results by subtracting $f(x)$ from both sides of equation to the left. There is therefore indeed an universal connection between values of $f(-x)$ and $f(x)$ - both are of same magnitude but with inverted signs. Minus sign can be therefore get 'pulled out' of the function argument⁴¹. In scientific language: our observed function $f(x)$ is uneven. (The fact that $f(0) = 0$ is in accordance with this.)

If a whole number n is negative, then there is a natural number m with $n = -m$; for random x we then get:

$$f(n \cdot x) = f(-m \cdot x) = f(m \cdot [-x]) = m \cdot f(-x) = m \cdot [-f(x)] = -m \cdot f(x) = n \cdot f(x)$$

All whole numbers can be therefore get pulled out of the function argument. (The case of factor $n = 0$ is automatically included through $f(0)$.)

Let $n \geq 1$ be a natural number, for any x the following is true:

$$f(x) = f(1 \cdot x) = f\left(\frac{n}{n} \cdot x\right) = f\left(n \cdot \frac{x}{n}\right) = n \cdot f\left(\frac{x}{n}\right) \Rightarrow f\left(\frac{x}{n}\right) = \frac{1}{n} \cdot f(x)$$

Logical conclusion on right side results by dividing both sides of equation to the left by n . Let λ be a random *rational* factor: $\lambda = p/q$ with whole number p and natural number q , then the following is true:

$$f(\lambda \cdot x) = f\left(\frac{p}{q} \cdot x\right) = f\left(p \cdot \frac{x}{q}\right) = p \cdot f\left(\frac{x}{q}\right) = p \cdot \frac{1}{q} \cdot f(x) = \frac{p}{q} \cdot f(x) = \lambda \cdot f(x)$$

Is it therefore possible to pull out *any* factor from the function argument!?

Wishful thinking! Not all numbers are rational, there are also the irrational ones. Hmm...

Back to the problem text. The function is described as being defined for the whole set of real numbers (good, because we always used condition of *any or arbitrary* value of x , by using it we are unable to restrict it) and *stable*. What are they good for?

Here is our answer! Let λ^* be an irrational factor (it could also be rational but in that case following explanation becomes futile); there exists *series* λ_k of *rational* numbers, that converge towards λ^* .

Let us for example set $\lambda^* = \sqrt{2}$; then with $\lambda_1 = 1, \lambda_2 = 1.4, \lambda_3 = 1.41, \lambda_4 = 1.414, \lambda_5 = 1.4142$ etc. we

⁴¹together with property provided in the problem text - it is very important! This is not valid for all functions, as for example $y = f(x) = \sqrt{x}$: here is $f(4) = 2$, but $f(-4)$ is not -2 , but rather doesn't exist.

choose values of series that with each index contain one decimal case more of the real value λ^* . Finite decimal fractions are *rational* numbers. If $k \rightarrow \infty$ it is also $\lambda_k \rightarrow \lambda^*$. In respect to index k set value of x , according to rules of *limits of series* we get $\lambda_k x \rightarrow \lambda^* x$, and because of the demanded stability of $f(x)$ it is therefore $f(\lambda_k x) \rightarrow f(\lambda^* x)$. For the same limit we have

$$\lim_{k \rightarrow \infty} f(\lambda_k x) = f(\lambda^* x)$$

and

$$\lim_{k \rightarrow \infty} f(\lambda_k x) = \lim_{k \rightarrow \infty} [\lambda_k \cdot f(x)] = f(x) \cdot \lim_{k \rightarrow \infty} \lambda_k = f(x) \cdot \lambda^*$$

and thus $f(\lambda^* x) = \lambda^* f(x)$ is universal. Now λ^* is indeed *any* real number. There is yet a small nuance to cover left - it is common knowledge that every number is a multiple of one and itself:

$$f(x) = f(x \cdot 1) = x \cdot f(1)$$

For a change once in a while we pulled x out of the function argument.

What was it that we were searching for? Oh right, an accurate description of $f(x)$. We have it now:

$$f(x) = f(1) \cdot x$$

What can be said about the factor $f(1)$? It is not mentioned in the problem text. It is arbitrary, let us name it c . (We can still nevertheless keep it in our mind that it is $f(1)$.) It makes our searched function $f(x) = cx$ - a linear function without y-intercept value, or one that goes through the origin. All functions of this form are the ones that fulfill the conditions of the problem.

Factor c is arbitrary, but set for a particular function $f(x)$.⁴²

Excited reader should now ask the question - did I understand this derivation?

Actually nothing was really complicated. If all equations were as simple as these.. One can only wonder with what simple means such fundamental conclusions can be achieved - thoughtful reasoning - a quality which gets little recognition in our contemporary fast-paced society.

Next question: can you put this text aside and repeat step-by-step solution off the top of your head? Those who have memorizing ability of memory artists could probably master this in short period of time but it is not what all of this is about. It is more important to develop understanding(or feel) for these steps with goal of using it in one form or another when solving new problems. One should be able to observe that solution consists of structures that are embedded in each other - bigger conclusions are made step-by-step through combining simple facts with previously made conclusions. Lets assume one does indeed undergo this effort(it is indeed one-no doubt about that). What does one get from it? Chosen function $f(x)$ has two properties, one primary and a secondary that is extracted from it. For all values following is valid:

$$f(x_1 + x_2) = f(x_1) + f(x_2) \text{ and } f(\lambda x) = \lambda f(x).$$

This pair of properties is being called **linearity**. It is something fundamental. Engineering students get to know this through the circumstance that they get introduced to systems of linear equations, linear differential equations etc. These methods are also popular in different fields of application. Direct, small but eventually crucial use of this observation is: when $f(x_1 + x_2) = f(x_1) + f(x_2)$ is valid in all cases, it means that $f(x)$ is of type $f(x) = cx$.

Counter-conclusion : If $f(x)$ is not of this type, then $f(x_1 + x_2) = f(x_1) + f(x_2)$ is invalid (or not valid for all pairs of values, eventually valid for some).⁴³

Simple fact: function $f(x) = \sqrt{x}$ is not of type cx , therefore $\sqrt{x_1 + x_2} = \sqrt{x_1} + \sqrt{x_2}$ is not valid for all (non-negative) x and one is not allowed to calculate

$$\int x\sqrt{4+x^2}dx = \int x(2+x)dx$$

like this.

Yeah, this stuff has its own consequences, and one odd to think about it.

By the way test subject⁴⁴ already knew it. He did calculate the length of vector $[2.79, 3.82]^T$ correctly as $l = \sqrt{2.79^2 + 3.82^2} = 4.73$ and not $l = 2.79 + 3.82$. Or maybe subjects inner voice warned him: *Be careful! If*

⁴²For more and universal information on this subject - see supplement→ theory→ linear mapping

⁴³This type of conclusion is called counter-position. It says that if B follows from A, then invalidity of A follows from invalidity of B. However invalidity of B does not follow from invalidity of A.

Everything clear?

All right, all right: Assume that bank is open Tuesdays and Thursdays until 6PM at these two days only (on other days there are other service hours). Therefore from A = 'today is Tuesday' follows B = 'bank is open till 6PM'. From not-A = 'today is not Tuesday' we do not get not-B = 'bank is not open till 6PM' - it could be Thursday. If one sees that bank is closed at 3PM then it means not-B, and therefore not-A: 'today is not Tuesday.'

⁴⁴Editors note: the one that made this mistake while integrating in one of W.ROSENHEINRICH'S exams

it would be so easy then they would write $l = x_1 + x_2$ ⁴⁵ instead of $l = \sqrt{x_1^2 + x_2^2}$ in formula handbooks.

Besides: $\sqrt{0+1} = \sqrt{0} + \sqrt{1}$ — sometimes it does get true!

One more example? $f(x) = 1/x = x^{-1}$ is also not of type cx , although it somewhat looks like one when examined superficially.

Consequence: it is not

$$\frac{1}{x_1 + x_2} = \frac{1}{x_1} + \frac{1}{x_2}$$

or in applied case:

$$\int \frac{dx}{x^2 + 3x + 2} = \int \frac{dx}{x^2} + \int \frac{dx}{3x + 2}$$

Finally I would like to mention that not every *linear function* $f(x) = mx + n$ has the property of *linearity* - $n = 0$ must also be true!

‘*Linear*’ therefore has several meanings, just like ‘*homogeneous*’ (which comes up in certain linear problems)⁴⁶

⁴⁵more accurate - as in little bit less false would be: $l = |x_1| + |x_2|$

⁴⁶One other severely overcrowded word would be ‘*order*’, as quoting DONALD E. KNUTH: ” *Since only two of our tape drives were in working order I was ordered to order more tape units in short order, in order to order the data several orders of magnitude faster.*”

B.) This is a clear problem only at first glance. If solution of it is ordered from outside one should not be ashamed to ask:

‘What does parallel mean?’

Mean reaction of the one who ordered it should diminish rapidly by realizing the fact that he does not know it for sure himself.

If one sets it as a task to solve on ones own in case of some project, than one needs to answer this by oneself. But in this case one is also likely to know how to interpret it.

For the start: if one of both vectors is a null-vector(or both) - are they then to be seen as parallel or not? It is a question of belief. It must get decided through arrangement and not through calculation. Maybe it is irrelevant as in this particular case both vectors are not null-vectors.

Careful programmers of type *burned kids* would not be sure of this and cover this case at first with the following cut through Gordian knot:

If $(|x_1| + |x_2| + |x_3|) \cdot (|y_1| + |y_2| + |y_3|) = 0$ then: *parallel*.

After this one can be sure that both of the vectors are not null-vectors. If one decides to spare this out then it is kind of risky - one does after all fly with single-engine planes too.

Now comes a more sophisticated question: are $[3, 1, 0]^T$ and $[1, 0.3333, 0]^T$ parallel? Strictly speaking - no. In common everyday sense-yes.

Computer has no use of such cunning human commentary. Its bird brain needs clear instructions - when are vectors parallel and when are they not? EUCLIDS observations about lines that do not intersect or intersect only at infinite distance are not suitable for it. It needs a clear calculation method. Not only it but also the programmer.

Theoretically, parallel would mean that all components of both vectors are in the same ratio to each other: $y_k/x_k = \lambda, k = 1, 2, 3$ with one and the same value λ (obviously excluding component pairs whose both values are zero).

It is however unreasonable to test if real type numbers are equal. One could for example check if division values are near to each other - in that case vectors are seen as being parallel. If $x_1 x_2 x_3 \neq 0$ (this must get checked as well - it leads to program having too many logic divisions) we can for example set $\lambda_k = y_k/x_k, k = 1, 2, 3$ and conclude that both vectors are parallel if:

$$\frac{|\lambda_1 - \lambda_2| + |\lambda_1 - \lambda_3|}{|\lambda_1| + |\lambda_2| + |\lambda_3|} < 10^{-6}.$$

is true. This idea is good only at first glance.

Vectors $[1, 0.000001, 2]^T$ and $[2, 0.000001, 4]^T$ would then not be seen as parallel, which contradicts our liberal interpretation.

I think that following arrangement would be reasonable: two (non-null) vectors can be called parallel when angle α between them is smaller than small angle of our choice ϵ or bigger than $\pi - \epsilon$.

It means that $|\cos \alpha| \geq \cos \epsilon$ or

$$\frac{|x_1 y_1 + x_2 y_2 + x_3 y_3|}{\sqrt{(x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2)}} \geq \cos \epsilon.$$

To avoid calculation of roots one can calculate both sides of the inequality squared:

$$\frac{(x_1 y_1 + x_2 y_2 + x_3 y_3)^2}{(x_1^2 + x_2^2 + x_3^2) \cdot (y_1^2 + y_2^2 + y_3^2)} \geq \cos^2 \epsilon = \frac{1 + \cos 2\epsilon}{2} \approx \frac{1 + (1 - 2\epsilon)^2}{2} = 1 - \epsilon^2$$

Since ϵ is a very narrow angle one can easily substitute its cosine(as well as that of double the angle) with beginning part of cosine power series. Resulting difference to the real value can be ignored.

Those are 3D vectors that we are dealing with so we can use cross-product to calculate angle α :

$$\frac{|\vec{x} \times \vec{y}|}{\sqrt{(x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2)}} = |\sin \alpha| \leq \sin \epsilon \approx \epsilon$$

With it the given problem is solved. It includes the null-vector test mentioned at the beginning and then only a single test instead of several different cases. The program should contain angle ϵ as variable argument.

Note: If one wants a universal program then the solution above wont do.

Assume that positive numbers can have values from $1.00 \cdot 10^{-99}$ till $9.99 \cdot 10^{+99}$ - vector \vec{x} with dimensions $x_1 = x_2 = x_3 = 10^{51}$ would crash the program in the same fashion as one with values $x_1 = x_2 = x_3 = 10^{-51}$. As precaution one could find out which component has the greatest value and then divide the corresponding vector by this value. Resulting rounding errors can be ignored. Lengths of vectors would in this case range between 1 and $\sqrt{3}$.